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THREE—DIMENSIONAL GEOMETRIC MOMENT SPACE BOUNDS
WITH APPLICATIONS TO PROBLEMS IN COMMUNICATION THEORY

M. A. KING, JR.

UCLA • SCHOOL OF ENGINEERING AND APPLIED SCIENCE

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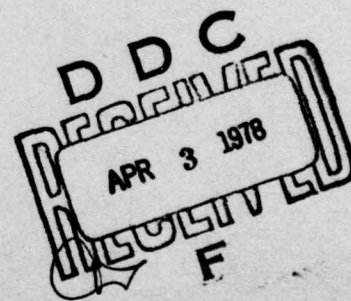
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The solution to many problems in communication theory takes the form of a moment of a function of a random variable. Often this moment is difficult to evaluate numerically. When this is the case, tight bounds to the true value are sought that are relatively easy to evaluate. One method of deriving such bounds is a geometrical technique that is a result of an Isomorphism Theorem from Game Theory. Recently very useful bounds have been derived with this technique using two-dimensional geometries.

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This report extends this work into a useful class of three-dimensional geometries. This class of geometrices can produce very tight bounds with a reasonable amount of effort.

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I.   INTRODUCTION

1.1   Problem Description

There are many important problems in the field of communications theory that have as their solution the expectation of a random variable. Perhaps the classic example of such a problem is that of computing the probability of bit error for a binary signal being transmitted on a channel with linear intersymbol interference [1] - [15]. A block diagram for this example is given in Figure 1.1.
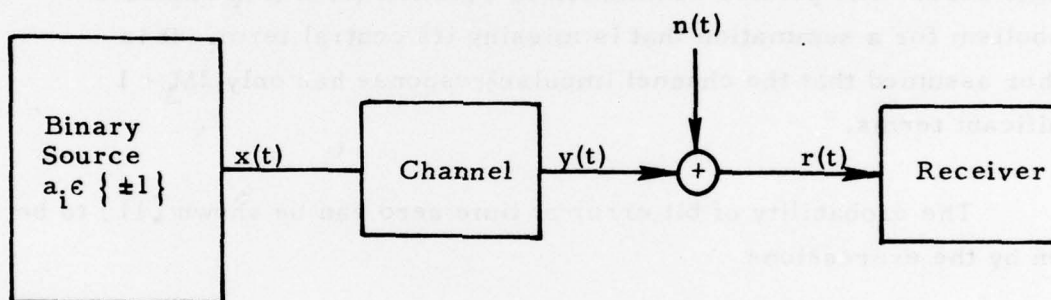
Figure 1.1

The binary source selects a value for a_i with equal probability each T seconds. These source symbols are encoded into waveforms suitable for transmission across the channel. The time function x(t) represents a string of these channel waveforms. The channel is assumed to act upon the waveform string x(t) as a linear filter. Thus, the waveform associated with a particular source symbol will typically be distorted in shape and spread in time by the action of the channel. Let the distorted waveform string at the channel output be represented by y(t). This signal is assumed to be further distorted by the addition of a white Gaussian noise process, that is denoted by n(t). The waveform string finally presented to the receiver is r(t) where

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$$r(t) = y(t) + n(t). \tag{1.1}$$

This signal is detected and sampled. The sampled output at time zero can be represented by

$$r_o = a_o h_o + \sum_{i=-M}^{M}{}' a_i h_i + n_o \tag{1.2}$$

where r_i is the detected and sampled output at time i, $\ldots a_{-M} \ldots a_{-1}$ $a_o a_1 \ldots a_M \ldots$ is the binary input signal string, $\{h_i\}$ is the sampled impulse response of the channel, and n_o is the Gaussian noise sample at time zero. The primed summation in equation (1.2) is a standard symbolism for a summation that is missing its central term. It is further assumed that the channel impulse response has only $2M + 1$ significant terms.

The probability of bit error at time zero can be shown [11] to be given by the expressions

$$P_e = E_U \left[Q \left(\frac{h_o + u}{\sigma} \right) \right] \tag{1.3a}$$

$$= E_U \left[\left(Q \left(\frac{h_o + |u|}{\sigma} \right) + Q \left(\frac{h_o - |u|}{\sigma} \right) \right) / 2 \right] \tag{1.3b}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-y^2/2)\, dy, \tag{1.4}$$

$$u = \sum_{i=-M}^{M}{}' a_i h_i, \tag{1.5}$$

σ is the standard deviation of the Gaussian noise, and U represents the space of all strings of 2M binary symbols. The expressions (1.3a) and (1.3b) are clearly equal mathematically, but it has been shown [11] that one form or the other can have analytical advantages when evaluating the probability.

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Unfortunately, the expressions in equations (1.3a) and (1.3b) may be difficult or computationaly impractical to solve exactly. For example, if the interymbol interference extends for forty samples preceeding and trailing the actual signal sample time (M=40), the exact evaluation of the probability of error would involve the summation of $2^{80} \approx 10^{24}$ terms of the form of equation (1.4). Thus, even if equation (1.4) could be solved in 1 nanosecond of computer time, exact computation of P_e would require 3×10^5 centuries. Although channels having an impulse response that is significant over 80 bit times may be rare, it is clear that the computation involved in calculating (1.3a) or (1.3b) can still be large even for fairly modest impulse responses.

Expressions similar to (1.3a) can be derived for the probability of bit error on any additive Gaussian noise channel with linear interference. Examples would include spread spectrum multiple access channels, and channels with co-channel interference [16] - [19]. Thus, the evaluation methods that will be discussed below are more generally applicable than to just intersymbol interference problems. They will apply to Gaussian channels with other kinds of linear interference as well.

1.2   Isomorphism Theorem

One possible approach to problems in communications and information theory that appear difficult or impossible to solve in their exact form is to find bounds to the exact solution that are easily computed. One technique that has been proven to be useful in providing bounds to problems of this kind is the Moment Bounding Technique [11] - [19]. This technique is based on an Isomorphism Theorem from Game Theory [20], [21]. In this approach, the moment of the function of the random variable that is of interest is bounded in terms of moments of other functions of the same random variable. These other functions are chosen such that their moments are relatively easy to evaluate. This approach has the unique advantage that both upper and lower bounds can be found with the same computational technique. In addition, the moment bounds that are derived using two or three dimensions yield a relatively simple geometrical understanding of the bounding process.

The Isomorphism Theorem can be stated as follows:

Isomorphism Theorem:

Let u be a random variable with probability distribution $G_U(u)$ defined over a finite closed interval $I = [a, b]$. Let $k_1(u)$, $k_2(u)$, ..., $k_n(u)$ be n continuous functions defined on I. Let m_i, $i = 1, \cdots, n$, denote the n generalized moments of the random variable u induced by the functions $\{k_i(u)\}$.

$$m_i = \int_I k_i(u) \, d\, G_U(u) = E_U[k_i(u)], \qquad i = 1, ..., n \qquad (1.6)$$

Denote the moment space $\mathcal{M}$ as

$$\mathcal{M} = \left\{ \underline{m} = (m_1, m_2, ..., m_n) \in R^n \right\} \qquad (1.7)$$

where $G_U(u)$ ranges over the set of all probability distribution functions defined on I. $\mathcal{M}$ is a closed, bounded, and convex set.

Let $\mathcal{C}$ denote the generalized curve $\underline{r} = (r_1, r_2, ..., r_n)$ traced out in R^n by $r_i = k_i(u)$ for $u \in I$. Let $\mathcal{H}$ be the convex hull of $\mathcal{C}$. Then $\mathcal{M} = \mathcal{H}$.

The application of the Isomorphism Theorem to bounding problems can be seen from the following two dimensional (n = 2) example. Given a function $k_1(u)$ of the random variable u whose moment, $E_U[k_1(u)]$, is desired, select a second function, $k_2(u)$, whose moment is easily computable. By identifying the functions $k_1(u)$ and $k_2(u)$ with the two orthogonal axes of a two-dimensional coordinate system, as in Figure 1.2, a curve $\mathcal{C}$ can be traced out as u varies through its finite range of values. The convex hull, $\mathcal{H}$, of the curve $\mathcal{C}$ can now be found, as in Figure 1.3. Let m_2 denote the value of the moment of the function $k_2(u)$ as in (1.6). According to the Isomorphism Theorem, the set of all moment pairs

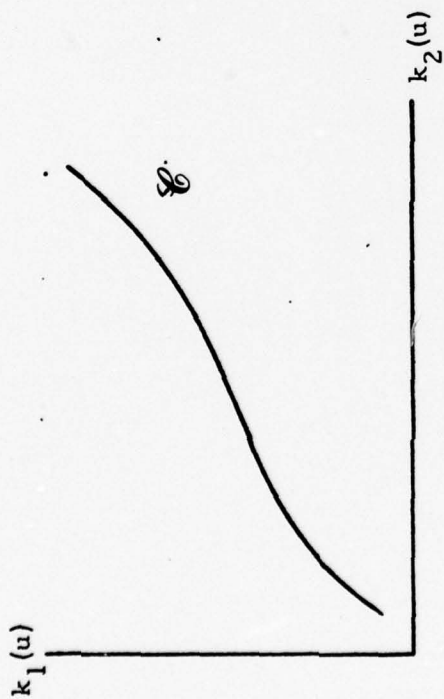$$\mathcal{M} = \left\{ (m_1, m_2) \mid m_1 = E[k_1(u)], \ m_2 = E[k_2(u)] \right\} \qquad (1.8)$$

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$k_2(u)$

m_2

Pe_U

Pe_L

$k_1(u)$

Figure 1.4

$k_2(u)$

$\mathcal{L}$

$k_1(u)$

Figure 1.2

$k_2(u)$

$\mathcal{H}$

$k_1(u)$
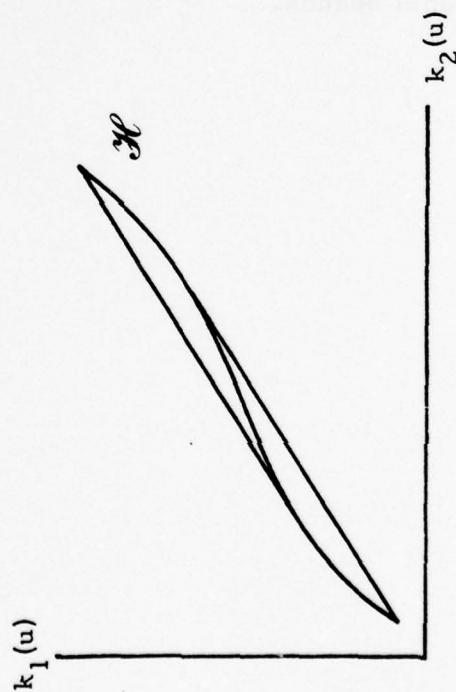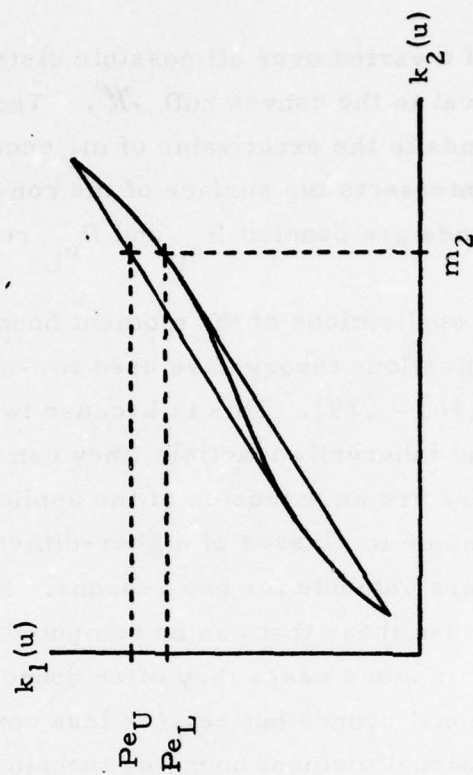
Figure 1.3

as the distribution of u varies over all possible distributions defined on the range of u, is identical to the convex hull $\mathcal{H}$.   Thus, from Figure 1.4, upper and lower bounds to the exact value of m_1 occur at the points where the line $k_2(u) = m_2$ intersects the surface of the convex hull. In Figure 1.4, the values of the bounds are denoted P_{e_U} and P_{e_L} respectively.

Most of the applications of the moment bounding techniques to problems in communications theory have used two-dimensional moment bounds [8] - [13], [16] - [19]. This is because two dimensional bounds are quite intuitive and inherently tractible (they can always be found graphically). The following chapters are an extension of the applications of the moment space bounding technique to classes of higher-dimensional bounds. Higher dimensional bounds are valuable for two reasons. First, they usually offer tighter bounds than those that can be computed with two-dimensional techniques. Second, in some cases they offer bounds that are as tight as the best two-dimensional bounds but require less computational effort. Thus a higher-dimensional moment bounding technique offers tigher bounds, *or less effort, or in some cases both tighter bounds and less effort than two-*dimensional bounds.

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II.   THREE-DIMENSIONAL MOMENT SPACE BOUNDS

2.1   Preliminary Definitions

A finite length twisted curve $\mathscr{C}$ can be defined in E^3 by three parametric equations:

$$x = h(u) \tag{2.1}$$
$$y = g(u) \tag{2.2}$$
$$z = f(u) \tag{2.3}$$

where $u \epsilon I_u = [a, b]$ and a and b are finite. The functions h, g, and f will be assumed to be smooth and single valued on I. By smooth it is meant that the functions are continuous and have continuous derivatives of all orders on $I'_u = (a, b)$, the interior of the region I_u. When the values x, y, and z are associated with the axes of a standard orthogonal right handed coordinate system, the curve $\mathscr{C}$ will be traced out in three dimensions as u covers its range of values, I_u.

An equivalent definition of the curve $\mathscr{C}$ is given by

$$y = y(x) = g(h^{-1}(x)) \tag{2.4}$$
$$z = z(x) = f(h^{-1}(x)) \tag{2.5}$$

where x takes values in the finite interval defined by $I_x = [h(a), h(b)]$.
The finiteness of I_x is assured by the finiteness of I_u and the smoothness of h. This second representation for the curve $\mathscr{C}$ will prove to be very useful in the development that follows.

It will be assumed that the z-axis is associated with the function whose moment is to be bounded. The x and y axes will be associated with the auxillary functions that have been selected to be used in the bounding.

A representative sketch of a curve $\mathscr{C}$ and the functions y(x) and z(x) is given in figure 2.1.

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The convex hull $\mathcal{H}$ of a finite length twisted curve $\mathcal{C}$, can be defined in two equivalent ways. The first could be termed an interior definition in that it defines $\mathcal{H}$ in terms of points known to belong to $\mathcal{H}$. These are the points on the curve $\mathcal{C}$.

Defn: The <u>convex hull</u>, $\mathcal{H}$, of a twisted curve $\mathcal{C}$, is the collection of all points p representable as linear convex combinations of points of $\mathcal{C}$. That is

$$p = \lambda_1 \, p_1 + \lambda_2 \, p_2 + (1 - \lambda_1 - \lambda_2) \, p_3 \tag{2.6}$$

$$p_1, \; p_2, \; p_3 \; \epsilon \, \mathcal{C}$$

$$\lambda_1, \; \lambda_2, \; (1 - \lambda_1 - \lambda_2) \geq 0$$

It can be shown [22] that the number of points in the convex combination (2.6), need never exceed the dimensionality of the curve $\mathcal{C}$.

The second definition of a convex hull generated by the curve $\mathcal{C}$ could be termed an exterior definition. This is because the definition involves the intersection of a sequence of closed spaces that converge to the hull from its exterior.

Defn: The <u>convex hull</u> $\mathcal{H}$, of a twisted curve $\mathcal{C}$, is equal to the intersection of the set of all closed half spaces that contain the curve $\mathcal{C}$.

A plane that defines a member of the set of half spaces, that also contains points of $\mathcal{C}$, will be called a tangent plane or a support plane to $\mathcal{H}$ at those points.

When using the Isomorphism Theorem to solve moment space bounding problems, it is the surface of the convex hull that is of interest. The surface can be thought of as being constructed of three kinds of features.

These are 1) isolated (or extreme) points, 2) chords, and 3) planar sections. This sort of division into features can easily be understood in terms of tangent planes to the surface of $\mathcal{H}$. It is clear that a tangent plane to a convex surface can have only three kinds of sets of points in common with the surface. The first kind is the single (isolated or extreme) point. The second would be a line segment (chord). This is clear because if any two points belong to a convex body, all points on the line segment connecting them must be in the convex body. The third, by simple extension of this argument, is a planar segment. When thought of in terms of tangent planes, the features are said to be the characteristics of the particular appropriate tangent planes to $\mathcal{H}$.

In general, deriving a mathematical description of the convex hull $\mathcal{H}$ generated by a given twisted curve $\mathcal{C}$ is a difficult problem. Therefore, the development presented below will deal with a class of curves $\mathcal{C}$, whose members are encountered in error probability problems, and whose convex hulls are easily characterized.

It will be assumed that the function $y(x)$ is strictly convex with respect to x on I_x the domain of y. This assumption is not seriously limiting in practice. This is because the functions x and y are the functions that have been selected to be used in the derivation of the bounds. They can easily be selected specifically such that they are convex with respect to each other.

This convexity assumption has two major implications. The first is that every point on the curve $\mathcal{C}$ is an isolated point on the surface of $\mathcal{H}$. The second is that the chord connecting the end points of $\mathcal{C}$ will lie on the surface of $\mathcal{H}$. The first implication is important because of the fact that the convex hull of a set of points is identical to the convex hull of the extreme or isolated points in the set [22], [23]. Thus, if the set of points is the curve $\mathcal{C}$, and every point of $\mathcal{C}$ is an isolated point, then every point of $\mathcal{C}$ is required to define the convex hull $\mathcal{H}$. Therefore, every point of $\mathcal{C}$ is on the surface of $\mathcal{H}$, and every point of $\mathcal{C}$ is required in the definition of $\mathcal{H}$.

The importance of the second implication is that the chord and the curve $\mathscr{C}$ form a closed loop that is entirely on the surface of the convex hull $\mathscr{H}$. This fact will be useful in proving the lemmas that follow.

Both of these stated implications can be varified with the help of Figure 2.1. It can be seen in Figure 2.1 that the convex hull of the curve $y(x)$ is bounded by the curve $y(x)$ itself, and the chord connecting the end points of $y(x)$. But by construction, $\mathbf{y}(x)$ is the x-y plane projection of the curve $\mathscr{C}$. Furthermore, it can be seen that the x-y plane projection of the convex hull $\mathscr{H}$ will be the convex hull of the x-y plane projection of $\mathscr{C}$, which is $y(x)$. But since the projection of the curve $\mathscr{C}$ and the projection of the chord joining the end points of $\mathscr{C}$ are the boundary of the projected hull, the original curve $\mathscr{C}$ and the original chord must lie on the boundary of the hull $\mathscr{H}$ [22].

Another idea that will be required in the following development is the notion of a regularity condtion.

<u>Defn.</u>   The surface of a convex hull, $\mathscr{H}$, generated by a finite length twisted curve $\mathscr{C}$, will be said to conform to a <u>regularity condition</u> if the surface is characterized entirely by chords that emanate from the endpoints of $\mathscr{C}$.

Curves that generate hulls that meet this regularity condition will be seen to yield bounds that are easily computable. This is because all points on the surface of these hulls can be described in terms of chords emanating from the endpoints of the curve $\mathscr{C}$. In particular, the points on the surface of $\mathscr{H}$ that correspond to the desired moment space bounds will be points of the chords that emanate from the endpoints of $\mathscr{C}$ and intercept the line

$$x = m_1 \tag{2.7a}$$

$$y(x) = m_2 \tag{2.7b}$$

There will be two such chords, corresponding to the two endpoints of $\mathscr{C}$. The points of interception of these chords with the line of equations (2.7a, b) will yield the desired upper and lower bounds.

When the regularity condition is met, the bounds can be evaluated as follows. Consider Figure 2.2. Figure 2.2 shows the x-y plane projection of the curve $\mathscr{C}$ and the two chords emanating from the endpoints of $\mathscr{C}$ denoted ℓ_1 and ℓ_2. In this projection, the line of equations (2.7a, b) shows as a point. Since two points define a line, the endpoints of $\mathscr{C}$, denoted (y_{min}, x_{min}) and (y_{max}, x_{max}) in Figure 2.2, and the point (m_1, m_2), completely define both the chord ℓ_1 and the chord ℓ_2. By using the definition of these chords and equation (2.4) the points $(y_1, x_1) = (y(x_1), x_1)$ and $(y_2, x_2) = (y(x_2), x_2)$ can be computed in terms of the values of x_1 and x_2.

$$\frac{y_{max} - y(x_1)}{x_{max} - x_1} = \frac{y_{max} - m_2}{x_{max} - m_1} \tag{2.8}$$

$$\frac{y(x_2) - y_{min}}{x_2 - x_{min}} = \frac{m_2 - y_{min}}{m_1 - x_{min}} \tag{2.9}$$

By solving equations (2.8) and (2.9) for x_1 and x_2 respectively, and using equation (2.5), the values of the bounds $z_o^{(1)}$ and $z_o^{(2)}$ may be computed

$$z_o^{(1)} = z_{max} - (z_{max} - z(x_1)) \frac{x_{max} - m_1}{x_{max} - x_1} \tag{2.10}$$

$$z_o^{(2)} = z_{min} + (z(x_2) - z_{min}) \frac{m_1 - x_{min}}{x_2 - x_{min}} \tag{2.11}$$

where

$$z_{max} = z(x_{max}), \tag{2.12}$$

and

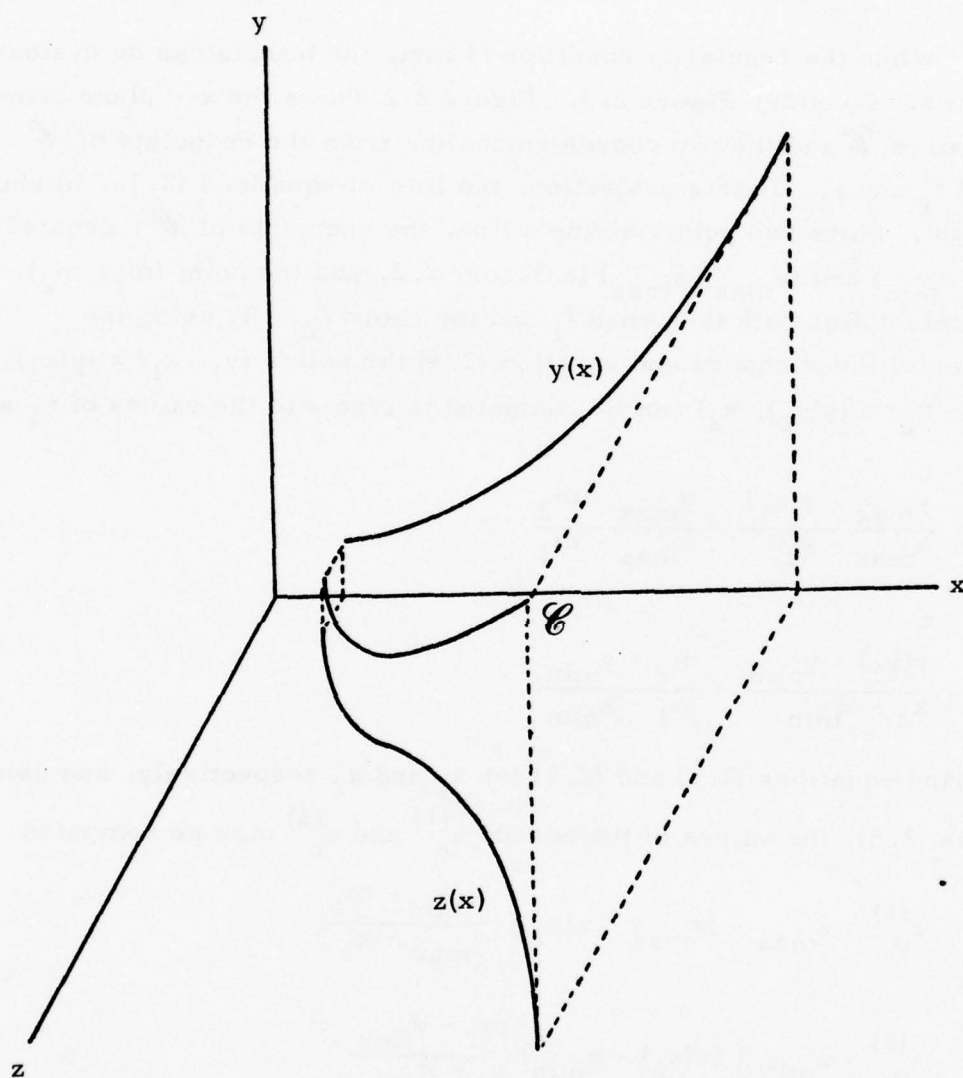$$z_{min} = z(x_{min}). \tag{2.13}$$

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Figure 2.1

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One of $z_o^{(1)}$ and $z_o^{(2)}$ will be the upper bound and the other will be the lower bound. The determination of whether the upper bound (for instance) has the form of equation (2.10) or equation (2.11) will depend on the nature of the functions $y(x)$ and $z(x)$.
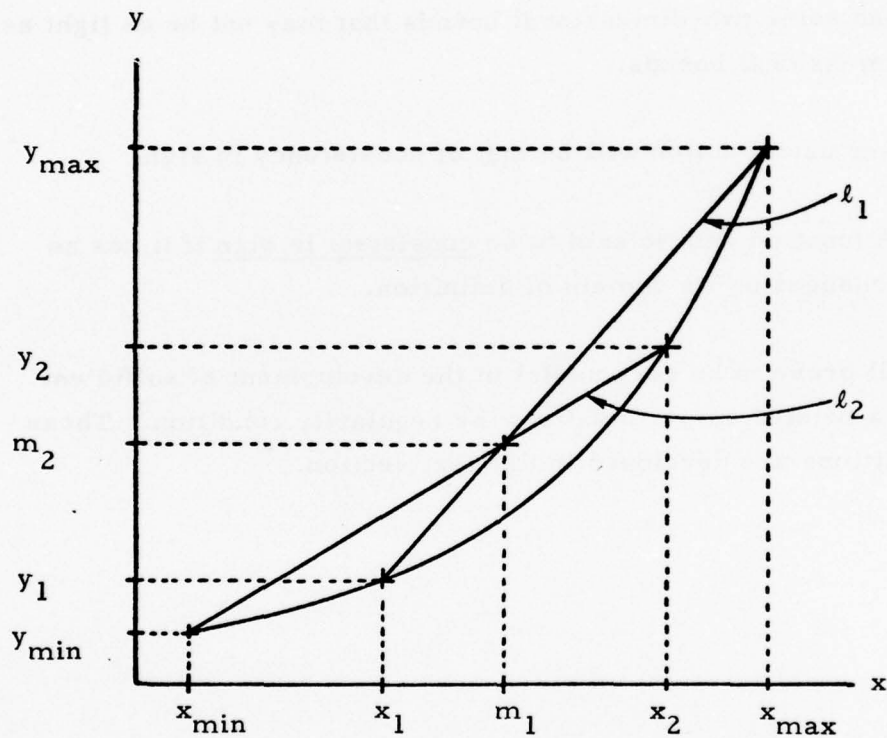
Figure 2.2

The bounds of equations (2.10) and (2.11) appear to have several important practical computational advantages. The first is that equations (2.10) and (2.11) are of a very simple form. The second is that the form of the bounding equations remains the same over the region where the regularity condition is satisifed. As long as this condition is satisifed, the form of the bounding equations is independent of any of the bounding parameters (e.g., the values of y_{max}, x_{max}, m_1, m_2, etc.). This has not been typically the case for the two-dimensional moment space bounding results that have been presented earlier [11], [12], [16] - [19]. Consequently, there bounds will be relatively easy to evaluate, possibly even easier than some two-dimensional bounds that may not be as tight as these three-dimensional bounds.

Another useful notion will be that of consistency in sign.

Defn: A function will be said to be consistent in sign if it has no zeros or sign changes on its domain of definition.

This notion will prove to be very useful in the development of sufficient conditions for a twisted curve to satisfy the regularity condition. These sufficient conditions are developed in the next section.

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2.2.   Sufficient Conditions

In this section, criteria are developed that insure that the regularity condition defined in the previous section holds for a particular twisted curve $\mathscr{C}$. The development of useful criteria will be presented as a sequence of four lemmas, and a discussion of these lemmas. The proofs of the lemmas are presented as appendicies.

Lemma 1:

Consider a chord, ℓ, connecting any two points of a twisted curve $\mathscr{C}$. Let the convex hull generated by $\mathscr{C}$ be denoted by $\mathscr{H}$. If any interior point of ℓ (not an endpoint) is on the surface of $\mathscr{H}$, the entire chord ℓ is on the surface of $\mathscr{H}$.

Proof:

Appendix 1.

This lemma states that if any interior point of a chord is on the surface of $\mathscr{H}$, the whole chord is on the surface. It can be seen that this implies that if any point of a chord can be shown to be other than a surface point of $\mathscr{H}$, then no interior point of the chord can be on the surface. This lemma is of use in discriminating between chords that are being investigated as possible surface features of some convex hull. Consider the sketch of Figure 2.3
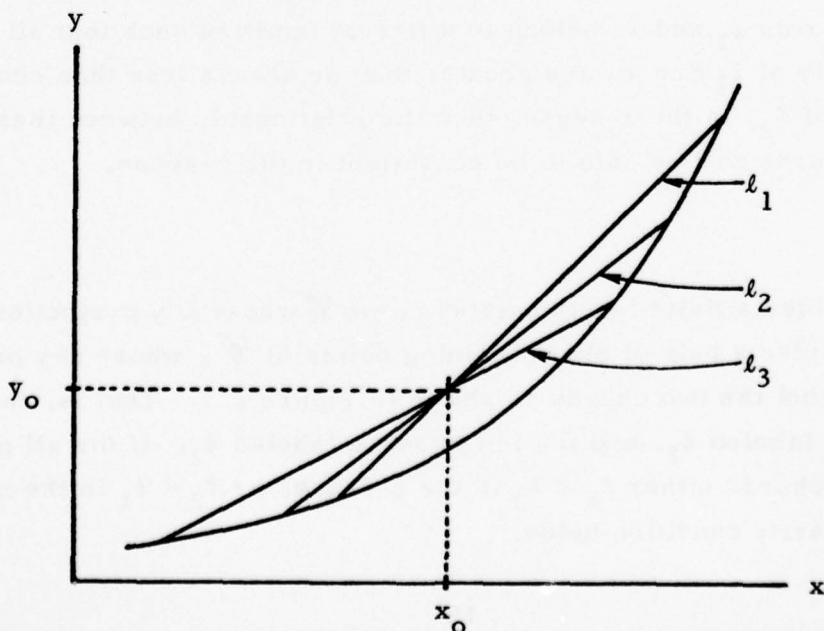

Figure 2.3

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Figure 2.3 represents the x-y plane projection of a finite length twisted curve $\mathscr{C}$ and three chords, ℓ_1, ℓ_2, and ℓ_3, joining points on $\mathscr{C}$. The point (x_o, y_o) is the projection of a line parallel to the z-axis in E^3. Such a line will intersect the surface of the convex hull generated by $\mathscr{C}$ in two points [22]. Assume that the chord ℓ_1 lies on the surface of the convex hull $\mathscr{H}$, generated by $\mathscr{C}$, and includes one of these two points. Similarly, assume ℓ_3 is also a surface chord and includes the other of the two points. Now, by Lemma 1, ℓ_2 can have no interior points on the surface of $\mathscr{H}$ unless either ℓ_1 and ℓ_2, or ℓ_3 and ℓ_2 intersect in E^3.

An important concept that will be needed in the following development is the notion of an ordering of chords in the z-sense. This idea is most easily understood pictorially. Thus, a sketch that outlines the ideas involved is given as Figure 2.4.

Given the curve $\mathscr{C}$, consider two chords, denoted ℓ_1 and ℓ_2 in Figure 2.4, whose x-y plane projections intersect. Let the x-coordinate of this point of intersection in the x-y plane be denoted by x_o. Clearly there is a point on each of the chords ℓ_1 and ℓ_2 in E^3 whose x-coordinate has the value x_o. Let the z-coordinate value of these points on the chords ℓ_1 and ℓ_2 be denoted by z_1 and z_2 respectively. Then, if as in Figure 2.4, $z_2 > z_1$, it will be said that chord ℓ_2 is greater than (>) chord ℓ_1 in the z-sense. If the ordering of the values of z_1 and z_2 had been reversed, the ordering of the chords in the z-sense would have been reversed. The concept of ordering in the z-sense combines naturally with the concept of consistency in sign. If chords ℓ_1 and ℓ_2 belong to different families such that all chords from the family of ℓ_1 are always greater than or always less than chords of the family of ℓ_2, in the z-sense, then the relationship between these families of chords will be said to be consistent in the z-sense.

Lemma 2:

Consider a finite length twisted curve $\mathscr{C}$ whose x-y projection is convex. Consider a pair of chords joining points of $\mathscr{C}$, whose x-y projections intersect. Label the two chords as shown in Figure 2.5. That is, the right-most chord is labeled ℓ_1, and the left-most is labeled ℓ_2. If for all possible pairs of such chords either $\ell_1 > \ell_2$ in the z-sense, or $\ell_2 > \ell_1$ in the z-sense, then the regularity condition holds.

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Ordering of Chords in the Z-Sense
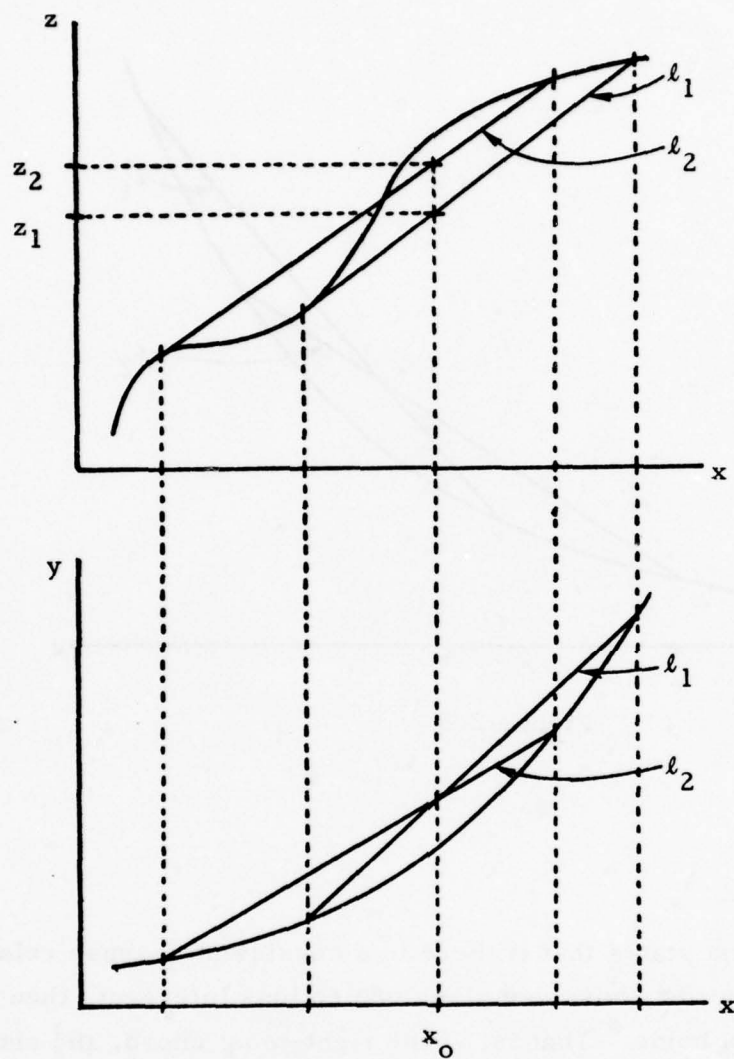
Chord ℓ_2 > Chord ℓ_1 in Z-Sense

Figure 2.4
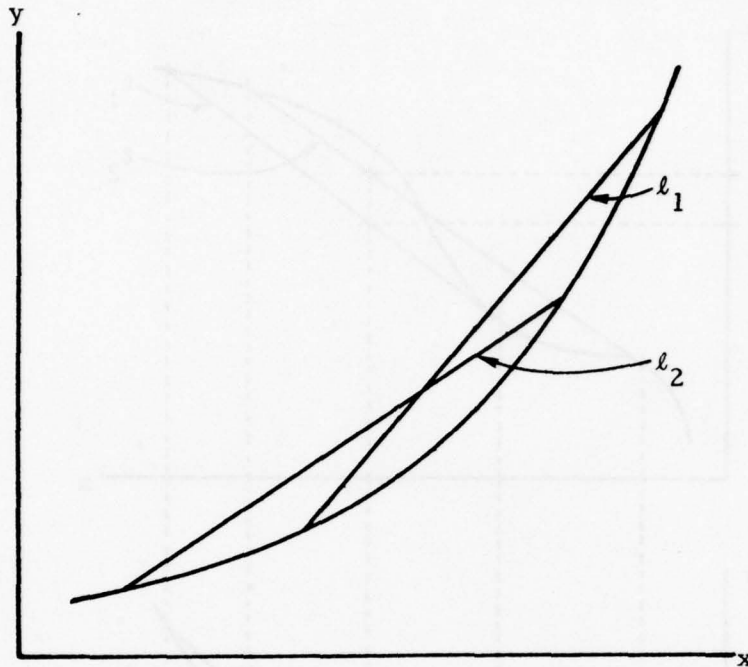
Figure 2.5

Proof:

    Appendix 2.

    This lemma states that if there is a consistent z-sense relationship between pairs of chords whose x-y plane projections intersect, then the regularity condition holds. That is, if the right-most chord, the chord ℓ_1 in Figure 2.5, is always above, in the z-sense a chord to the left of it, then the highest chord must be the one that is placed as far right as possible, i.e., the chord that terminates on the right endpoint of the curve $\mathscr{C}$. But then the lowest chord must be placed as far left as possible, i.e., the chord that emanates from the left end point of $\mathscr{C}$. The conclusion is similar if chord ℓ_1 is always lower instead of always higher, but the positions are reversed. In either case the highest and lowest chords emanate or terminate on endpoints of $\mathscr{C}$. These chords must all be on either the upper or lower surface. This is exactly the definition of the regularity condition.

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Lemma 2 is a valid test for the regularity condition. However, it would be very difficult to use Lemma 2 as a criterion for testing for the regularity condition because of its non-constructive nature. Lemmas 3 and 4 deal with the problem of finding a criterion that is more constructive.

<u>Lemma 3:</u>

Consider a twisted curve $\mathscr{C}$, and chords ℓ_1 and ℓ_2 as in Lemma 2. Let the endpoints of the chords be labeled as shown in Figure 2.6
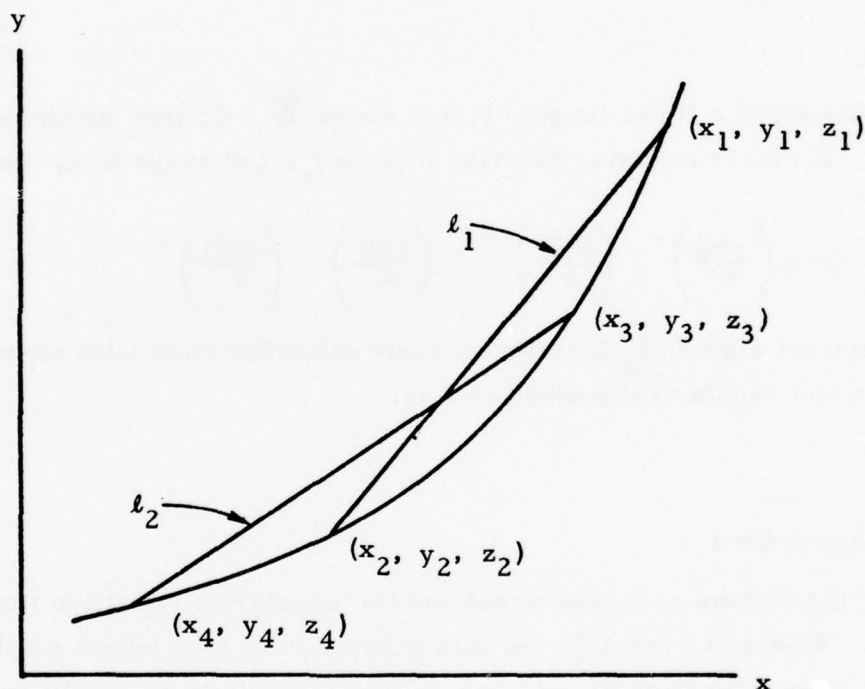
Figure 2.6

For each particular chord pair similar to that of Figure 2.6, relabel the axes to force the point (x_4, y_4, z_4) to the origin. Then if for all such possible chord pairs, the term

$$\left(\frac{y_3}{x_3} - \frac{y_2}{x_2} \right) \left(\frac{z_1}{x_1} - \frac{z_2}{x_2} \right) - \left(\frac{y_1}{x_1} - \frac{y_2}{x_2} \right) \left(\frac{z_3}{x_3} - \frac{z_2}{x_2} \right) \qquad (2.14)$$

is of consistent sign, the regularity condition holds.

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Proof:

Appendix 3.

Lemma 3 is an improvement over Lemma 2 in the sense that Lemma 3 provides a more constructive test for the regularity condition. However, this test involves making computations based on all sets of four points on the curve $\mathscr{C}$, and therefore, is still not practical. The shifting of the point (x_4, y_4, z_4) to the origin can be seen to reduce the complexity of the term (2.14), but has no other effects. Thus, a more constructive test is needed. Such a test is given by Lemma 4.

Lemma 4:

Consider a finite length twisted curve $\mathscr{C}$, defined parametrically in x by y(x). If y(x) is a convex function of x on I_x, the range of x, and

$$V(x) = \left(\frac{y(x)}{x}\right)'' \left(\frac{z(x)}{x}\right)' - \left(\frac{z(x)}{x}\right)'' \left(\frac{y(x)}{x}\right)' \tag{2.15}$$

is of consistent sign on I_x (primes indicate differentiation with respect to x), then the regularity condition holds.

Proof:

Appendix 4

This lemma provides a test for the regularity condition that is tractable. Where the term (2.14) was a function of four points on the curve $\mathscr{C}$, (2.5) is a function of only one. Thus (2.15) can be easily evaluated on I_x to determine whether or not the conditions of Lemma 4 are met. This evaluation will be seen to be especially easy for the special case of Corollaries 4.1 and 4.2.

Corollary 4.1:

Consider a finite length twisted curve $\mathscr{C}$ defined as in Lemma 4. If $y(x) = x^2 + ax$ for any real number a, and

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$$V_1(x) = \left(\frac{z(x)}{x} \right)''$$

(2. 16)

is of consistent sign on I_x, then the regularity condition holds.

For the important special case of the function $y(x)$ being quadratic in x, the requirement that $\left(\frac{z(x)}{x} \right)$ be strictly convex is sufficient to insure that the regularity condition holds. This is an extremely easy condition to verify. The class of functions x and $y(x)$ related as in Corollary 4.1 is important because it includes functions that often have relatively easily computed moments. Particular examples include the second and fourth moments and exponential moments (i. e., $E_x[e^{bx}]$).

Occasionally, however, it is cumbersome to determine whether or not the function $\left(\frac{z(x)}{x} \right)''$ is of consistent sign. In such cases, Corollary 4.2 offers a somewhat more restrictive, but perhaps more easily computed criterion.

Corollary 4.2:

If for the twisted curve $\mathscr{C}$ as defined in Corollary 4.1 equation (2.16) is replaced with

$$V_2(x) = z'''(x)$$

(2. 17)

and $V_2(x)$ is of consistent sign on I_x, then the regularity condition holds.

The results leading to Lemma 4 can be formulated as a heorem. This theorem summarizes the bounding results that have been obtained thus far.

Theorem 1:

Consider the convex hull $\mathscr{H}$, generated by a finite length twisted curve $\mathscr{C}$ that satisfies the condition of Lemma 4. Then either

A) $y(x)$ is convex $\cup$

or

B) $\mathbf{y}(x)$ is convex $\cap$

and either

C) $V(x)$ (equation (2.15)) is consistently positive

or

D) $V(x)$ is consistently negative.

If A) and D) are true, or B) and C) are true, then the upper bound will be determined by a chord emanating from the endpoint of $\mathscr{C}$ associated with the maximum value of x.   The lower bound will be determined by a chord emanating from the end-point of $\mathscr{C}$ associated with the minimum value of x. This is to say that the upper bound can be evaluated using equation (2.10), and the lower bound evaluated using equation (2.11).   If A) and C) are true, or if B) and D) are true, then the roles of the chords and equations (2.10) and (2.11) are reversed.

An explicit proof of Theorem 1 will not be presented.   The essential arguments are included in the proofs of Lemma 3 and 4.   A proof of Theorem 1 would amount to keeping track of the sign of the consistent sign terms as they are developed in Appendicies 3 and 4.   Presentation of such a bookkeeping proof would be tedious and unenlightening.

2.3 Examples of Numerical Results

The theoretically derived bounding results that are stated in Theorem 1 will be applied to two specific examples. Both examples are intersymbol interference problems of the type discussed earlier. The major difference between the two examples is the choice of auxiliary functions used in obtaining the bounds.

Example A:

The first example to be considered utilizes exponential functions for both the $h(u)$ and $g(u)$ functions of equations (2.1) and (2.2). In this example the curve $\mathscr{C}$ is given by the three parametric equations

$$x = h(u) = e^{c(h_o + u)} \tag{2.18}$$

$$y = g(u) = e^{2c(h_o + u)} \tag{2.19}$$

$$z = f(u) = Q\left(\frac{h_o + u}{\sigma}\right) \tag{2.20}$$

where c is an arbitrary constant whose value will be selected later, and all other terms are as defined in Section I.1. It is clear from equations (2.18) and (2.19) that

$$y = x^2. \tag{2.21}$$

Thus, Corollaries 4.1 and 4.2 apply. The moments of the functions x and y were derived in [11] to be

$$m_x = E_U\left[e^{c(h_o + u)}\right] = e^{ch_o} \prod_{i=-M}^{M}{}' COSH(ch_i) = e^{ch_o} m'_x \tag{2.22}$$

$$m_y = E_U\left[e^{2c(h_o + u)}\right] = e^{2ch_o} \prod_{i=-M}^{M}{}' COSH(2ch_i) = e^{2ch_o} m'_y \tag{2.23}$$

where

$$m'_x = \prod_{i=-M}^{M}{}' COSH(ch_i) \tag{2.24}$$

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$$m'_y = \prod_{i=-M}^{M} {}' \; COSH(2ch_i).$$ (2.25)

It will be further assumed that the intersymbol interference "eye is open." This is a reference to an intuitive method of measuring the amount of intersymbol interference with an oscilloscope. Quantitatively, it means that the amplitude of the desired signal is larger than the worst case interference.

$$h_o > D = \sum_{i=-M}^{M} {}' \; |h_i|$$ (2.26)

Qualitatively, this means that for a noise-free channel, perfect reception could be achieved by a memoryless receiver. In practice, if the "open-eye" assumption is not valid, the channel will produce so much signal distortion that the channel is probably not of interest.

By using equations (2.18) - (2.23) in equations (2.8) - (2.11) and simplifying the results, the equations for the bounds for this specific example are

$$\hat{m}_z = z_o^{(1)} = Q\left(\frac{h_o + \hat{u}}{\sigma}\right) \frac{m'_x - e^{-cD}}{e^{c\hat{u}} - e^{-cD}} + Q\left(\frac{h_o - D}{\sigma}\right) \frac{e^{c\hat{u}} - m'_x}{e^{c\hat{u}} - e^{cD}}$$ (2.27)

where

$$e^{c\hat{u}} = \frac{m'_y - m'_x e^{-cD}}{m'_x - e^{-cD}}$$ (2.28)

and

$$\check{m}_z = z_o^{(2)} = Q\left(\frac{h_o + \check{u}}{\sigma}\right) \frac{e^{cD} - m'_x}{e^{cD} - e^{c\check{u}}} + Q\left(\frac{h_o + D}{\sigma}\right) \frac{m'_x - e^{c\check{u}}}{e^{cD} - e^{c\check{u}}}$$ (2.29)

where

$$e^{c\check{u}} = \frac{m'_y - m'_x e^{cD}}{m'_x - e^{cD}}$$ (2.30)

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If the regularity condition holds, either equation (2.27) or equation (2.29) will represent the upper bound, and the other one will be the lower bound. Thus, it remains to determine if the regularity condition holds.

As mentioned above, equation (2.21) implies that Corollary 4.2 is applicable. The term $z''(x)$ may be computed from equation (2.18) and (2.20) using the chain rule of calculus. This term may be **expressed** as a function of the parameter u as

$$\frac{d^3 z}{dx^3}(u) = \frac{1}{c^3 \sigma \sqrt{2\pi}} \left[\frac{1}{\sigma^2} - \left(\frac{h_o + u}{\sigma^2} + c \right) \left(\frac{h_o + u}{\sigma^2} + 2c \right) \right] e^{-(h_o + u)^2/2\sigma^2 - 3c(h_o + u)}$$

(2.31)

where the parameter takes values in the region

$$-D \leq u \leq D < h_o .$$

(2.32)

According to Corollary 4.2, the regularity condition will hold if the right hand side (RHS) of equation (2.31) is consistent in sign. It can be seen that this depends only upon the term in square brackets,

$$\frac{1}{\sigma^2} - \left(\frac{h_o + u}{\sigma^2} + c \right) \left(\frac{h_o + u}{\sigma^2} + 2c \right).$$

(2.33)

This term must be consistent in sign for all values of the parameter u in [-D, D]. An analysis of (2.33) indicates that if the constant c lies in any one of three regions on the real line, consistency in sign is assured. These three regions are

Region I:

$$-\frac{3}{4} \frac{(h_o - D)}{\sigma^2} - \frac{1}{4} \sqrt{\frac{(h_o - D)^2}{\sigma^4} + \frac{8}{\sigma^2}} \leq c \leq -\frac{3}{4} \frac{(h_o + D)}{\sigma^2} + \frac{1}{4} \sqrt{\frac{(h_o + D)^2}{\sigma^2} + \frac{8}{\sigma^2}}$$

(2.34)

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Region II:

$$c \gtrsim -\frac{3}{4} \frac{(h_o - D)}{\sigma^2} + \frac{1}{4} \sqrt{\frac{(h_o - D)^2}{\sigma^4} + \frac{8}{\sigma^2}}$$ (2.35)

Region III:

$$c \leq -\frac{3}{4} \frac{(h_o + D)}{\sigma^2} - \frac{1}{4} \sqrt{\frac{(h_o + D)^2}{\sigma^4} + \frac{8}{\sigma^2}}$$ (2.36)

Thus, whenever the arbitrary constant c is chosen to lie in one of these three regions, the regularity condition will hold and the bounds of equations (2.27) and (2.29) are valid.

For purposes of computation, the bounds of equations (2.27) and (2.29) were evaluated at the four finite boundary points of the three regions. From among these four evaluations, the best was selected and included as an entry in the column headed "selected 3-D" in Table 2.1. For comparison, the optimum two dimensional bounds derived in [11] are presented as the column labeled "optimum 2-D."

The two-dimensional results presented in Table 2.1 were computed using the planar curve given by equations (2.18) and (2.20). The results are optimum in the sense that the values of c that yield the smallest upper bound and greatest lower bound have been computed. These computations amount to iterative solutions of a complicated nonlinear equation. This must be done twice, once for the upper and once for the lower bound, for each bounding point computed. Thus the values present in Table 2.1 required 10 iterative solutions.

The bounds of Table 2.1 are an example of three-dimensional bound offering results that are essentially equivalent to the best two-dimensional results, but require less computational effort. The three-dimensional bounds of Table 2.1, required the evaluation of eight explicit expressions. Each bound (upper and lower) was computed for each of the four values of the parameter c. Although the evaluation of equations (2.24) and (2.25) might be tedious if M is large, this task could easily have been accomplished with a standard scientific hand calculator. The optimal two-dimensional bounds required a general purpose computer for their evaluation.

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Chebyshev Channel:

$$h_i = 0.4023 \; COS(2.839|i| - 0.7553) \; EXP(-0.4587|i|)$$
$$+ \, 0.7162 \; COS(1.176|i| - 0.1602) \; EXP(-1.107|i|)$$

	LOWER BOUNDS		UPPER BOUNDS	
SNR (dB)	OPTIMUM 2-D	SELECTED 3-D	OPT 2-D	SELECTED 3-D
0	1.593×10^{-1}	1.593×10^{-1}	1.594×10^{-1}	1.593×10^{-1}
4	5.761×10^{-2}	5.765×10^{-2}	5.783×10^{-2}	5.779×10^{-2}
8	6.69×10^{-3}	6.68×10^{-3}	6.82×10^{-3}	6.82×10^{-3}
12	6.29×10^{-5}	6.08×10^{-5}	6.84×10^{-5}	7.15×10^{-5}
16	3.58×10^{-9}	2.86×10^{-9}	4.44×10^{-9}	5.55×10^{-9}

TABLE 2.1

Example B:

The second example also deals with the intersymbol interference problem, but in this case the bounds are derived using the second and fourth moment of the interference. In this case the curve $\mathcal{C}$ is given by the parametric equations

$$x = h(u) = u^2 \tag{2.37}$$

$$y = g(u) = u^4 \tag{2.38}$$

$$z = f(u) = \frac{1}{2}\left[Q\left(\frac{h_o + u}{\sigma}\right) + Q\left(\frac{h_o - u}{\sigma}\right)\right]. \tag{2.39}$$

Equation (2.39) has been written in this form so that the range of the variable u can be limited to the positive real axis.

$$o \leq u \leq D = \sum_{i=-M}^{M}{}' \; |h_i| < h_o \tag{2.40}$$

As is implied in expression (2.40), the "open eye" assumption is made for this example also. The moments associated with the functions (2.37) and (2.38) have been computed by Yan [12]. The form of these moments is

$$m_x = E_U[u^2] = \sum_{i=-M}^{M}{}' \; h_i^2 \tag{2.41}$$

$$m_y = E_U[u^4] = \left(\sum_{i=-M}^{M}{}' \; h_i^2\right)^2 + 4\sum_{i=-M}^{M}{}' \; h_i^2 \sum_{j=i+1}^{M}{}' \; h_j^2 \;. \tag{2.42}$$

If the regularity condition holds, the bounding equations can be derived from equations (2.10) and (2.11). These equations are

$$\hat{m}_z = \frac{m_x - u_o^2}{D^2 - u_o^2} \; W(D) + \left(1 - \frac{m_x - u_o^2}{D^2 - u_o^2}\right) \; W(u_o) \tag{2.43}$$

where

$$W(u) = Q\left(\frac{h_o + u}{\sigma}\right) + Q\left(\frac{h_o - u}{\sigma}\right) \tag{2.44}$$

and

$$u_o = \frac{D^2 m_x - m_y}{D^2 - m_x} \tag{2.45}$$

and

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$$\breve{m}_z = \frac{m_x^2}{m_y} \; W \left(\sqrt{\frac{m_y}{m_x}} \right) + \left(1 - \frac{m_x^2}{m_y} \right) \; W(0) \,. \tag{2.46}$$

It can be seen that these equations will be even easier to evaluate than those of the previous example.

Since $y(x) = x^2$, Corollary 4.1 may be invoked to say that if

$$V_1(x) = \frac{d^2}{dx^2} \left(\frac{z(x)}{x} \right) \tag{2.47}$$

is of consistent sign, then the regularity condition will hold for this example. Unfortunately, (2.47) will not be of consistent sign for all cases of interest. Figure 2.7 is a graph that shows the regions where the regularity condition holds. The abscissa of Figure 2.7 is the signal to noise ratio $\left(\frac{h_o}{\sigma} \right)$ in dB. The ordinate is the ratio of the maximum distortion, D, to the desired signal amplitude h_o. The boundries of the cross-hatched regions are the zeros of (2.47). For any particular signal to noise ratio, the regularity condition will hold if the maximum distortion to signal amplitude ratio $\left(\frac{D}{h_o} \right)$ is below the boundary. It is seen from Figure 2.7 that the regularity condition holds throughout the greatest part of the region of interest. It is somewhat disappointing that the regularity condition does not hold in the vicinity of 9 dB signal to noise ratio.

In spite of the lack of universal application of the regularity condition for this example, the bounds of equations (2.43) and (2.46) are very tight where they apply. Table 2.2 shows these bounds as applied to the Chebyshev channel that was exhibited in the previous example. The three-dimensional bounds are in the columns headed "3-D." The columns headed "2nd" and "4th" are the results of two-dimensional bounding using equations (2.37) and (2.39), and (2.38) and (2.39), respectively to define planar curves. These values were computed by Yan [12].

Comparing the three-dimensional results of Table 2.2 with the optimum two-dimensional results of Table 2.1, it is seen that for low to moderate signal to noise ratios, the three-dimensional results are tighter. This is a case where a simply computed three-dimensional bound offers better performance than a more computationally complicated two-dimensional bound.

Figure 2.7

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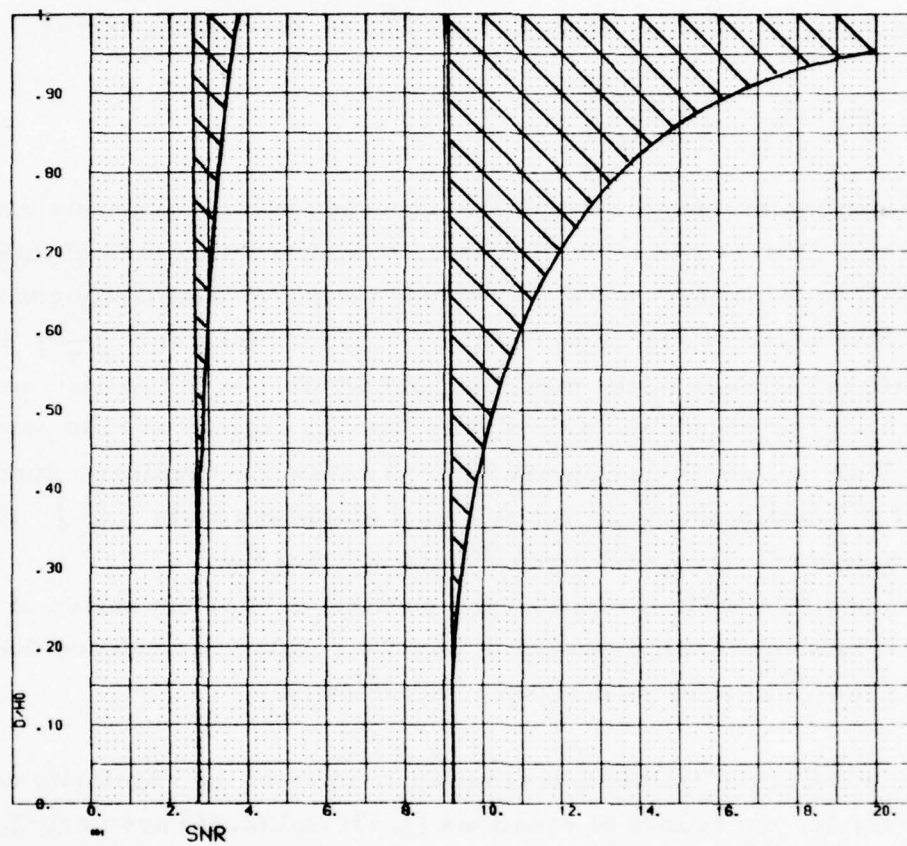
SNR (dB)	LOWER BOUNDS			UPPER BOUNDS		
	2nd	4th	3-D	2nd	4th	3-D
0	1.593×10^{-1}	1.588×10^{-1}	1.593×10^{-1}	1.593×10^{-1}	1.598×10^{-1}	1.593×10^{-1}
4	5.772×10^{-2}	5.673×10^{-2}	5.772×10^{-2}	5.773×10^{-2}	5.856×10^{-2}	5.772×10^{-2}
8	6.75×10^{-3}	6.16×10^{-3}	6.76×10^{-3}	6.84×10^{-3}	7.25×10^{-3}	6.76×10^{-3}
12	6.14×10^{-5}	4.78×10^{-5}	6.59×10^{-5}	1.05×10^{-4}	8.25×10^{-5}	6.73×10^{-5}
16	1.28×10^{-9}	2.83×10^{-9}	3.03×10^{-9}	1.04×10^{-7}	1.99×10^{-8}	1.66×10^{-8}

TABLE 2.2

2.4   Conclusions

The concept of the regularity condition has been shown to yield tight and easily computable bounds in cases of interest to communications engineers. In the first of two ISI examples, adjustment of the parameter c was shown to insure that the regularity condition held for all values of distortion and signal to noise ratio. The bounds on error probability could then be computed explicity from equations (2.27) and (2.29). In the second ISI example, the regularity condition is shown to hold for a subset of all cases of interest. However, the bounds were shown to be very tight for cases where the regularity condition held.

There are several advantages to this new three-dimensional technique over the two-dimensional bounding techniques. First, the three-dimensional bounds offer more possibilities of trading-off the tightness of the bounds and the difficulty of the computations than were previously available. This was illustrated in example A. Second, the best three-dimensional bounds will be inherently tighter than two-dimensional bounds. The consideration of an additional dimension can only add information to the approximation, and thereby tighten the resulting bounds.

However, the regularity condition bounds are somewhat limited in their applicability. As was shown in example A, there are restrictions on the value that the parameter c can take and still allow the condition to hold. This means that the optimum value of c cannot be used in general, because it may lie outside one of the allowed regions. Also, in example B it was found that there were certain values of signal-to-noise ratio and distortion that generated curves that could not be shown to meet the regularity condition. Thus, some attention had to be paid to these parameters in order to assure that the results of Table 2.2 were valid bounds.

Clearly there is some need for more general results. In particular, it would be useful to extend the region of known bounding results for the preceding two examples. A general method that will aid in obtaining these sorts of answers is presented in the next section.

III.   TANGENT PLANE METHOD

3.1   Introduction

          It was noted in Section 2.1 that there are two equivalent ways
to define a convex hull. These are termed the interior and the exterior
methods of definition. The interior method defines the hull as a convex
combination of points known to be in the hull. In the present context,
this means that the hull is defined in terms of convex combinations of
points belonging to the curve $\mathscr{C}$. The regularity condition results of
the previous chapter were based on this interior definition of the convex
hull.

          The exterior definition of the hull defines the hull in terms of
planes that are tangent to its surface. These tangent planes are also called
support planes [22]. They are planes that contain at least one point of the
hull, and have the property that they divide the space E^3 such that all of the hull
is contained in the plane and one of the two subspaces defined by the
plane. The point, or set of points that a tangent plane and the hull have
in common is called the characteristic of the tangent plane with respect to
the hull.

          The exterior method of hull definition can be used to generate
a second method of determining the surface characteristics of a convex hull.
This second method will be called the tangent plane method. This method
consists of a set of necessary conditions for when a chord or a planar
section can be the characteristic of a tangent plane. Clearly, all tangent
plane characteristics must be on the surface of the hull.

          A vital part of these new necessary conditions is the notion of
a tangent line to a twisted curve $\mathscr{C}$ in E^3 at a point p of the curve.

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Defn. [24]:

Given two points p and p_1 on a twisted curve $\mathscr{C}$, the <u>tangent line</u> to $\mathscr{C}$ at p is the limiting position of the line passing through the points p and p_1 as $p_1 \rightarrow p$ on the curve $\mathscr{C}$.

This notion of a tangent line to a twisted curve is the straight forward extension of the usual notion of a tangent line to a planar curve. For a curve $\mathscr{C}$ defined by x, y(x), and z(x), the tangent line is given by

$$X - x = \frac{Y-y}{y'} = \frac{Z-z}{z'} \tag{3.1}$$

where the capital letters are the coordinates of a generalized point on the line, the lower case letters are the coordinates of the point p, and the primes indicate differentiation with respect to x. It can be seen that the usual tangent to a planar curve (in the x-y plane) is given by the first equation of (3.1).

3.2   Fundamental Result

In this section, the basis of the tangent plane method is presented and discussed. This result is presented as Lemma 5.

Lemma 5:

Given a twisted curve $\mathscr{C}$ in E^3, and the convex hull $\mathscr{H}$ generated by $\mathscr{C}$, consider a tangent plane to $\mathscr{H}$ whose characteristic with respect to $\mathscr{H}$ is a chord, to be denoted Ch, joining points p_1 and p_2 of $\mathscr{C}$. This tangent plane must include the tangent lines to $\mathscr{C}$ at both of the points p_1 and p_2.

Proof: Appendix 5

This lemma provides a necessary condition for a chord joining points of a twisted curve $\mathscr{C}$ to be a chord on the surface of the convex hull generated by $\mathscr{C}$. The condition is that the plane defined by the chord Ch and one of the tangent lines, must be the same as the plane defined by Ch and the other tangent line. The three lines, Ch and both tangent lines, must all be in a single plane. This required relationship can be stated in the form of an equation.

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Lemma 6:

A necessary condition for a chord joining points p_1 and p_2 of a twisted curve $\mathscr{C}$, to be on the surface of the convex hull $\mathscr{H}$ generated by $\mathscr{C}$, is that the points $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ satisfy

$$\frac{y_1 - y_2}{x_1 - x_2}(z_1' - z_2') - \frac{z_1 - z_2}{x_1 - x_2}(y_1' - y_2') + (y_1' z_2' - y_2' z_1') = 0 \qquad (3.2)$$

where the primes denote differentiation with respect to x.

Proof: Appendix 6

The equation (3.2) is the direct mathematical consequence of the statement of lemma 5. It is a form of the mathematical relationship that must exist if the chord and the two tangent lines are to be co-planar.

For the important special case where $y(x) = x^2 + ax$, for any real number a, Corollary 6.1 is valid.

Corollary 6.1:

If the curve of Lemma 6 is such that $y(x) = x^2 + ax$, for any real number a, then the necessary relation (3.2) reduces to

$$\frac{z_1' + z_2'}{2} - \frac{z_1 - z_2}{x_1 - x_2} = 0 \qquad (3.3)$$

An interesting point of connection between the tangent plane method and the regularity condition results can be seen when the conditions are considered for when no pair of interior points of the curve $\mathscr{C}$ can meet the requirements of Corollary 6.1.

Corollary 6.2

For the conditions of Corollary 6.1, a sufficient condition that the equation (3.3) is never satisifed on the interior of the curve $\mathscr{C}$, is that the term $\left(\frac{z(x)}{x}\right)''$ be of consistent sign.

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The statement of Corollary 6.2 would seem to imply that if the term $\left(\frac{z(x)}{x}\right)''$ is of consistent sign, there can be no chords that are on the surface of the convex hull $\mathcal{H}$. This is because no chord joining two interior points of $\mathcal{C}$ could meet the necessary condition for being a surface chord. There clearly must be chords on the surface of every nondegenerate three-dimensional convex hull, however. The solution to this apparent contradiction is that the curve $\mathcal{C}$ is of finite length. The derivatives in equation (3.3) are not uniquely defined at the endpoints of $\mathcal{C}$. This means that equation (3.3) can be satisfied only by chords that emanate from one of the endpoints of $\mathcal{C}$. This is a surface that meets the definition of the regularity condition. This is consistent with Corollary 4.2.

Lemma 1 and Corollary 6.1 can be combined to produce a result that will help identify short surface chords. From Lemma 1 and the discussion immediately following it (Section 2.1) it is known that two chords joining points of the twisted curve $\mathcal{C}$, that do not intersect in E^3, but whose x-y plane projections intersect in E^2, as is illustrated in Figure 2.4, cannot both be chords that define the surface of the convex hull $\mathcal{H}$, generated by $\mathcal{C}$, in the same vacinity. That is, if both chords are on the surface of the hull, then they must be on opposite sides of the hull in the sense of the z-coordinate direction. This means that two chords that define adjacent points on the surface of the convex hull must be "nearly parallel" in some sense. This is because their x-y plane projections cannot intersect inside the projected boundries of the hull. Thus, if a chord that is on the surface of $\mathcal{H}$ joins two points of $\mathcal{C}$ whose x-axis coordinate values are x_1 and x_2 (assume without loss of generality that $x_2 > x_1$), then there must be a family of surface chords joining points whose coordinate values are denoted x_3 and x_4 ($x_4 > x_3$) such that $x_3 \geq x_1$ and $x_4 \leq x_2$. This is to say

$$d(x_1, x_2) = |x_2 - x_1| \geq d(x_3, x_4) = |x_4 - x_3.|$$ (3.4)

But these chords can be arranged as a sequence in order of decreasing values of $d(\cdot, \cdot)$. Since $\mathscr{C}$ was assumed to have continuous derivatives of all orders on I_x (section 2.1), $\mathscr{C}$ can have no linear regions. Therefore, the value of $d(\cdot, \cdot)$ in the sequence will approach zero as a limit. But this means that there is a point x_o that is spanned by all members of this family of chords. That is, for any member of the family of chords, let the endpoint x-axis coordinates be denoted x_1 and x_2 as before. Then $x_1 \leq x_o \leq x_2$. Let the point x_o be called the central point of this family of chords. The location of all possible central points for the twisted curve of Corollary 6.1 is given by Corollary 6.3.

Corollary 6.3:

For the finite length twisted curve $\mathscr{C}$ of Corollary 6.1, the locations of all possible central points are the solutions of the equation

$$z(x)''' = 0 \tag{3.5}$$

3.3 Necessary Conditions for Surface Planar Sections

The convex hull $\mathcal{H}$, generated by a finite length twisted curve $\mathcal{C}$ in E^3 will have three knids of surface characteristics. These characteristics were introduced in Section 2.1. They are 1) extreme (isolated) points, 2) chords, and 3) planar sections. It was pointed out in Section 2.1 that the assumed convexity of the coordinate function $y(x)$ on I_x implies that all extreme points of $\mathcal{H}$ will be points on the curve $\mathcal{C}$. Therefore, all remaining portions of the surface of $\mathcal{H}$ will be characterized by chords and planar sections. In the cases where the regularity condition was satisfied, the surface was made up entirely of chords. In this section, necessary conditions are established for when a planar section may be a surface characteristic of a convex hull. These necessary conditions are useful in the more general bounding problems when the regularity condition is not satisfied.

Lemma 7:

A necessary condition for a planar section defined by the points $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, and $p_3 = (x_3, y_3, z_3)$ of the twisted curve $\mathcal{C}$ to be a surface characteristic of the convex hull $\mathcal{H}$ generated by $\mathcal{C}$, is that the following equations be satisfied simultaneously.

$$\begin{bmatrix} 1 & y_1' & z_1' \\ 1 & y_2' & z_2' \\ 1 & y_3' & z_3' \end{bmatrix} \cdot \begin{bmatrix} y_3(z_1 - z_2) - z_3(y_1 - y_2) + (y_1 z_2 - y_2 z_1) \\ -x_3(z_1 - z_2) + z_3(x_1 - x_2) - (x_1 y_2 - x_2 z_1) \\ x_3(y_1 - y_2) - y_3(x_1 - x_2) + (x_1 y_2 - x_2 y_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.6}$$

$$(x_1 - x_2)(y_2' z_1' - y_1' z_2') - (y_1 - y_2)(z_1' - z_2') + (z_1 - z_2)(y_1' - y_2') = 0 \tag{3.7}$$

$$(x_1 - x_3)(y_3' z_1' - y_1' z_3') - (y_1 - y_3)(z_1' - z_3') + (z_1 - z_3)(y_1' - y_3') = 0 \tag{3.8}$$

$$(x_2 - x_3)(y_3' z_2' - y_2' z_3') - (y_2 - y_3)(z_2' - z_3') + (z_2 - z_3)(y_2' - y_3') = 0 \tag{3.9}$$

where the primes denote differentiation with respect to x.

Proof: Appendix 7

These six equations are the direct consequences of the necessary conditions for surface chords that were presented in Lemma 6. In fact, equations (3.7) - (3.9) can be seen to be the three versions of equation (3.2) of Lemma 6 that are obtained by selecting all combinations of two of the three points p_1, p_2, and p_3 and substituting them into equation (3.2). The matrix equation (3.6) is a consequence of the requirement that the tangent line to $\mathscr{C}$ at each of the three points p_1, p_2, and p_3, must be in the plane defined by p_1, p_2, and p_3.

For the special case where $y(x) = x^2 + ax$ for any real number a, the equations (3.6) - (3.9) become much simpler in appearence.

Corollary 7.1

If for the conditions of Lemma 7 the curve $\mathscr{C}$ is such that $y(x) = x^2 + ax$ for any real number a, then the equations for the necessary condition can be rewritten as:

$$(x_1 - x_3) \frac{z_1 - z_2}{x_1 - x_2} - (x_1 - x_2) \frac{z_1 - z_3}{x_1 - x_3} - z_1'(x_2 - x_3) = 0 \qquad (3.10)$$

$$(x_2 - x_3) \frac{z_1 - z_2}{x_1 - x_2} + (x_1 - x_2) \frac{z_2 - z_3}{x_2 - x_3} - z_2'(x_1 - x_3) = 0 \qquad (3.11)$$

$$(x_1 - x_3) \frac{z_2 - z_3}{x_2 - x_3} - (x_2 - x_3) \frac{z_1 - z_3}{x_1 - x_3} - z_3'(x_1 - x_2) = 0 \qquad (3.12)$$

$$\frac{z_1' + z_2'}{2} - \frac{z_1 - z_2}{x_1 - x_2} = 0 \qquad (3.13)$$

$$\frac{z_1' + z_3'}{2} - \frac{z_1 - z_3}{x_2 - x_3} = 0 \qquad (3.14)$$

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$$\frac{z_2' + z_3'}{2} - \frac{z_2 - z_3}{x_2 - x_3} = 0. \qquad\qquad (3.15)$$

Equations (3.10) - (3.15) were obtained by direct substitution of the relation $y_i = y(x_i) = x_i^2 + ax_i$, $i = 1, 2, 3$, into equations (3.6) - (3.9), and then simplifying. It can be seen that equation (3.6) and equations (3.10) - (3.12) are relationships between all three points p_1, p_2, and p_3. These equations define the necessary relationship between the three points and the tangent line to the curve $\mathscr{C}$ at one of them. The equations (3.7) - (3.9) and (3.13) - (3.15) are relations between combinations of two of the three points. These equations define the necessary relationships between the two tangent lines to $\mathscr{C}$ at two of the points and the chord that connects them. In order for the necessary conditions to be met, three points of $\mathscr{C}$ must be found that satisfy all six equations simultaneously. This fact leads directly to Lemma 8.

Lemma 8:

Given a finite length twisted curve $\mathscr{C}$, defined parametrically in x by y(x) and z(x), if $y(x) = x^2 + ax$ for any real number a, then there is no set of three interior points of $\mathscr{C}$ that satisfy Corollary 7.1.

Proof: Appendix 8

Lemma 8 implies that for the twisted curve $\mathscr{C}$ of Corollary 7.1, if there are any planar sections on the surface of $\mathscr{H}$, the convex hull generated by $\mathscr{C}$, then at least one and possibly two of the three points that define the planar sections must be an endpoint of $\mathscr{C}$. Thus, for the special case where $y(x) = x^2 + ax$, if the regularity condition holds, all surface features are chords and all the surface chords emanate from an endpoint of $\mathscr{C}$. If the regularity condition does not hold, all surface planar sections have an endpoint of $\mathscr{C}$ as one of their points of definition. Therefore, even when the regularity condition does not hold, this class of curves will be much easier to characterize than the general class of smooth twisted curves.

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The possibilities for surface planar sections that remain under the restrictions of Lemma 8 divide naturally into two classes. Class one contains the planar sections defined by two interior points and one endpoint. Class two includes the planar sections that are defined by a single interior point and both endpoints. Necessary conditions for these two classes of planar sections can be derived from the necessary conditions for chords in Corollary 7.1.

Considering class two first, the class of planar sections that are defined by a single interior point and two endpoints, the necessary condition is stated as Lemma 9.

Lemma 9:

Consider the twisted curve $\mathscr{C}$ of Corollary 7.1. Let the endpoints of $\mathscr{C}$ be denoted by $p_1 = (x_1, y_1, z_1)$ and $p_3 = (x_3, y_3, z_3)$. Let $p_2 = (x_2, y_2, z_2)$ denote an interior point of $\mathscr{C}$. Relabel the axis (if necessary) such that $p_1 = (x_1, y_1, z_1) = (0, 0, 0)$. Then a necessary condition for a class two planar section defined by p_1, p_2, and p_3, to be a surface characteristic of the convex hull generated by $\mathscr{C}$ is that

$$(z_2/x_2)' = \frac{(z_2/x_2) - (z_3/x_3)}{x_2 - x_3} . \tag{3.16}$$

Proof: Appendix 9.

Lemma 9 provides an easily usable test for determining the possible locations of class two surface planar sections. This test is perhaps most easily applied graphically. Consider the function $\rho(x) = z(x)/x$. Plot the function $\rho(x)$ as a function of x on I_x. An example of such a plot is given as Figure 3.1.
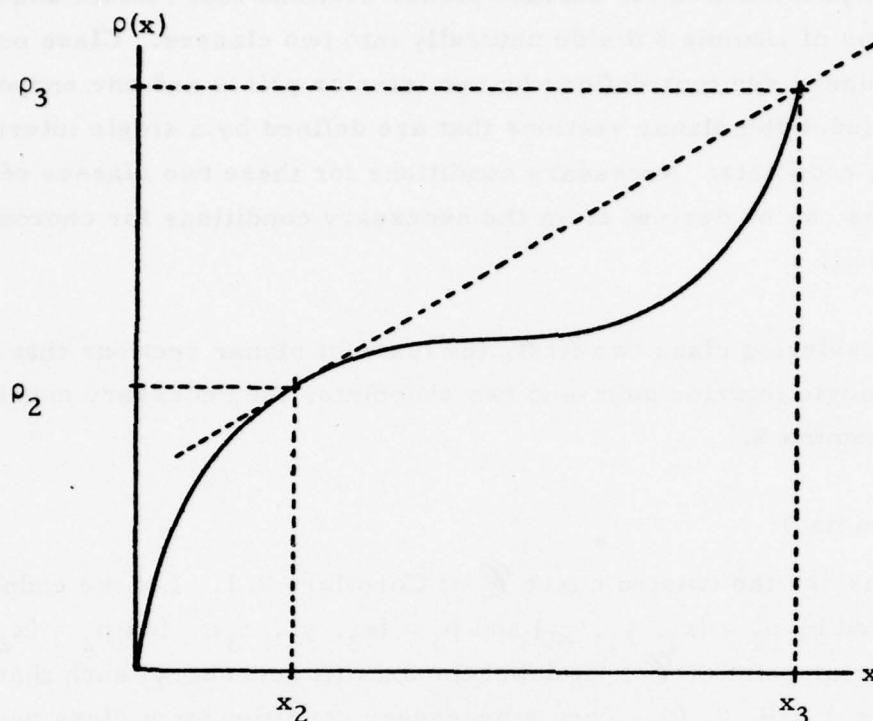
Figure 3.1

Equation (3.16) is satisfied when the tangent to the (two-dimensional) curve $\rho(x)$ passes through the right endpoint of $\rho(x)$. By construction, the right endpoint is the point $(x_3, \rho(x_3) = \rho_3)$. In Figure 3.1, the point labeled (x_2, ρ_2) satisfies this condition. It can be seen that this is the only point in the figure that satisifes the condition. Thus, if there is a class two planar section that is a surface characteristic of the convex hull generated by the curve $\mathscr{C}$, its interior point must be $(x_2, z(x_2))$, or a point similarly discovered by moving the endpoint p_3 to the origin. Thus, the numbers of possible planar sections that need be investigated is reduced to a small number.

A criterion for determining the possible locations of class one planar sections is presented as Lemma 10.

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Lemma 10:

For the twisted curve $\mathscr{C}$ of Corollary 7.1, denote an endpoint of $\mathscr{C}$ by $p_1 = (x_1, y_1, z_1)$. Let $p_2 = (x_2, y_2, z_2)$ and $p_3 = (x_3, y_3, z_3)$ be interior points of $\mathscr{C}$. Relabel the axis (if necessary) such that $p_1 = (x_1, y_1, z_1) = (0, 0, 0)$. A necessary condition for a class one planar section defined by p_1, p_2, and p_3, to be a surface characteristic of the convex hull generated by $\mathscr{C}$ is that

$$(z_3/x_3)' = (z_2/x_2)' = \frac{(z_2/x_2) - (z_3/x_3)}{x_2 - x_3} . \tag{3.17}$$

Proof: Appendix 10.

Using the notation $\rho(x) = z(x)/x$ as before, equation (3.17) can be rewritten as

$$\rho_3' = \rho_2' = \frac{\rho_2 - \rho_3}{x_2 - x_3} . \tag{3.18}$$

It is clear from (3.18) that the same kind of graphical analysis that was applied in the discussion of Lemma 9 will be useful with Lemma 10. Consider the plot of Figure 3.2. Figure 3.2 is an example of a plot of $\rho(x)$ verses x on I_x. The circumstances are identical to those of Figure 3.1. The shape of the plot is different for purposes of illustration.
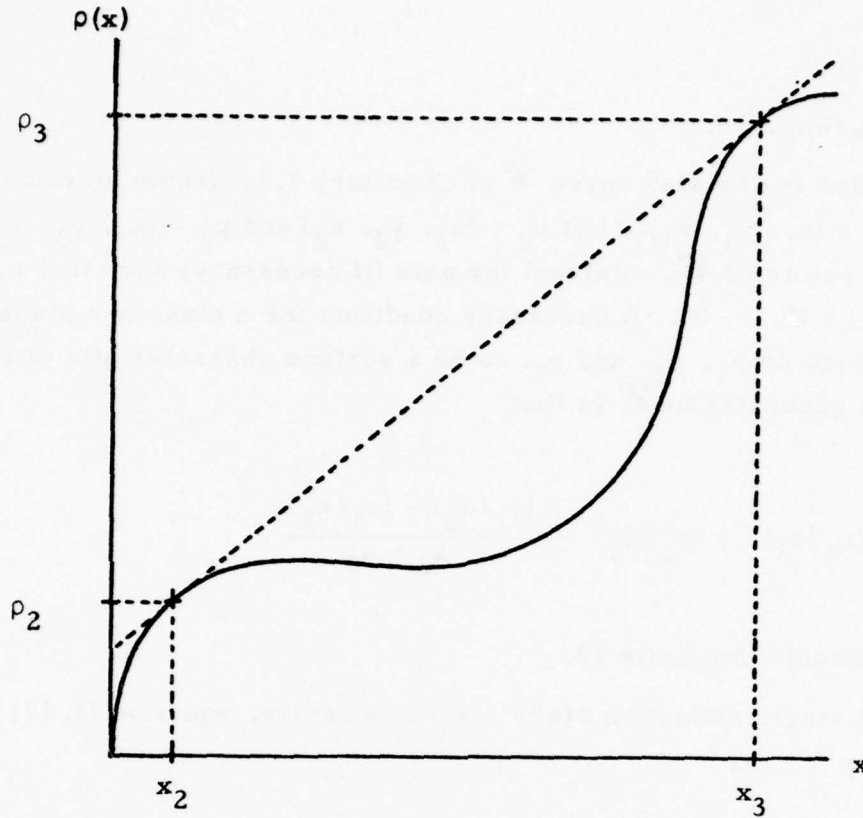
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Figure 3.2

From equation (3.18) it is seen that the pairs of points (p_2, p_3) that satisfy
equation (3.17) are pairs that share both the same value of slope and the
same tangent line to the curve in Figure 3.2. Such a pair of points is
labeled in Figure 3.2. Thus, if a class one planar section that emanates from
p_1, the endpoint of $\mathscr{C}$ that was shifted to the origin, is a surface feature
of the convex hull generated by $\mathscr{C}$, the points of $\mathscr{C}$ whose x-axis
coordinate values are x_2 and x_3 must be the interior points of the class
one planar section. That is, the planar section must be defined by the three
points $(0, 0, 0)$, $(x_2, y(x_2), z(x_2))$, and $(x_3, y(x_3), z(x_3))$. It is noted
that for both classes of planar sections, both endpoints must be shifted
to the origin in their turn. This is required to determine the full set of
possible surface planar sections.

Appendix 1

Proof of Lemma 1:

Let p denote the interior point of the chord ℓ that is on the surface of $\mathcal{H}$. Since p is on the surface, it must be part of the characteristic of some tangent plane T, to the surface of $\mathcal{H}$. Consider the relationship of this tangent plane T to the chord ℓ. Since the point p is in the plane T, either all of the chord ℓ is in T, or the chord ℓ penetrates T at the point p. If the chord ℓ is in T, the chord must be part of the characteristic of T on the surface of $\mathcal{H}$, and therefore, ℓ is on the surface of $\mathcal{H}$. If ℓ penetrates T, since ℓ joins two points of $\mathcal{C}$, there must be points of $\mathcal{C}$ (and hence of $\mathcal{H}$) in both half spaces defined by the plane T. This contradicts the definition of a tangent plane to the surface of $\mathcal{H}$. Therefore, ℓ must be in the tangent plane, and on the surface of $\mathcal{H}$ if any interior point of ℓ is on the surface of $\mathcal{H}$.

Q.E.D.

Appendix 2

Proof of Lemma 2:

Consider the family of chords joining points of the finite length twisted curve $\mathscr{C}$, whose x-y plane projections pass through the point (m_1, m_2). Three such chords are shown in Figure A2-1.
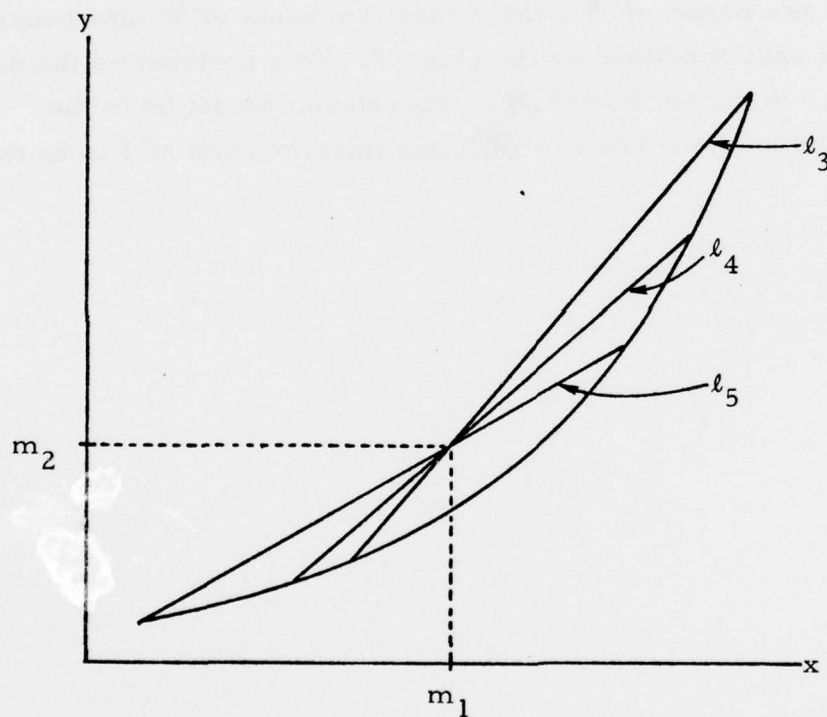

Figure A2-1

In Figure A2-1, chord ℓ_3 emanates from the upper endpoint of the x-y projection of the curve $\mathscr{C}$, the chord ℓ_5 emanates from the lower end point, and chord ℓ_4 is any other member of the family of chords whose projections pass through the point (m_1, m_2).

Consider the case where $\ell_1 > \ell_2$ in the z-sense for all chords as in Figure 2.5. Clearly from Figure A2-1, $\ell_3 > \ell_4 > \ell_5$ in the z-sense. In fact, the chord ℓ_3 must be the highest chord (in the z-sense) whose projection passes through (m_1, m_2). Since the curve $\mathscr{C}$ is convex in the x-y projection by assumption, the notions of upper and lower surfaces may be defined.

<u>Defn</u>: The curve $\mathscr{C}$ and the chord connecting the endpoints of $\mathscr{C}$ form a closed loop that lies entirely on the surface of $\mathscr{H}$, the convex hull generated by $\mathscr{C}$. This loop may be thought of as a dividing line that separates the surface of $\mathscr{H}$ into two surfaces. These surfaces will be termed the <u>upper and lower surfaces of the convex hull $\mathscr{H}$</u>, in terms of their relative positions with respect to the Z-axis. The surface of $\mathscr{H}$ can be considered to be the union of this upper and lower surface.

According to this definition, the upper surface of $\mathscr{H}$ at the point with x-y coordinates (m_1, m_2) is defined by the chord ℓ_3. But from Lemma 1, since this one interior point of chord ℓ_3 is on the surface, all of ℓ_3 is on the surface. But (m_1, m_2) is a general point on the x-y projection of the convex hull $\mathscr{H}$, defined by the twisted curve $\mathscr{C}$. Thus, for this case, the whole upper surface of the hull $\mathscr{H}$ is defined by chords similar to ℓ_3 -- chords that emanate from the upper endpoint of $\mathscr{C}$. Similar reasoning shows that the entire lower surface of $\mathscr{H}$ will be defined by chords similar to ℓ_5. Furthermore, identical reasoning will show that if $\ell_2 > \ell_1$ in the z-sense, then the surfaces formed by ℓ_3 and ℓ_5 will be reversed, but there will be no other effect. Thus, for the conditions of Lemma 2, both the upper and lower surfaces of $\mathscr{H}$ are defined by chords emanating from the end points of $\mathscr{C}$. By definition, this is the regularity condition. QED

Appendix 3

Proof of Lemma 3:

Consider the curve $\mathscr{C}$ and chord pair of Figure 2.6. Clearly, relabeling the axes to force the point $(x_4,\ y_4,\ z_4)$ to the origin will do nothing to the shape of $\mathscr{C}$ or the relative positions of the chords. The only advantage is that the formulas and intermediate mathematics are made simpler. The x-y plane projection of the curve and chord pair after the relabeling is shown in Figure A3-1.
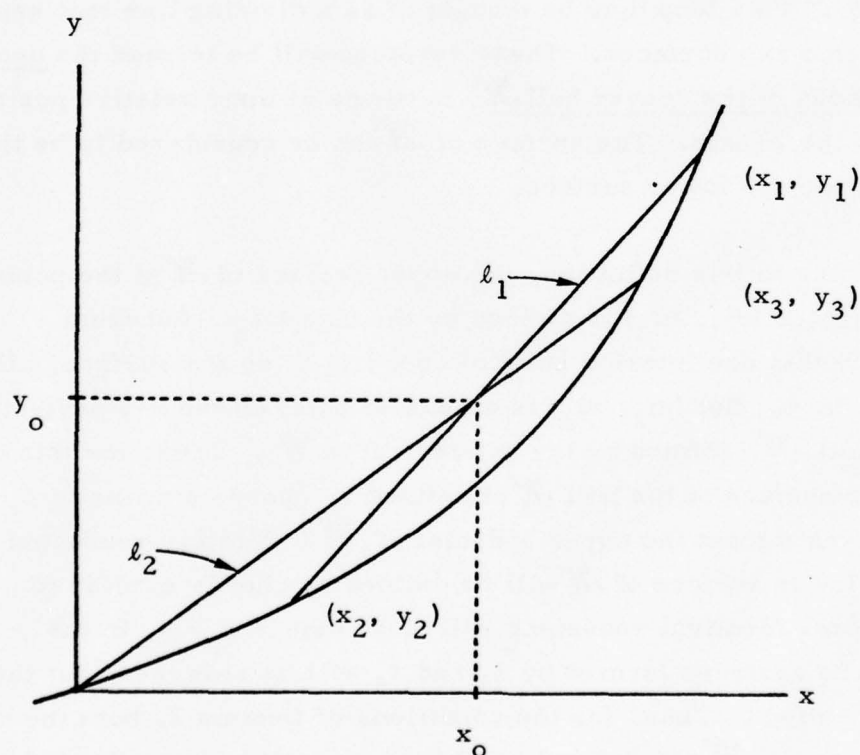

Figure A3-1

Label the point of intersection of the projections of the chords by (x_o, y_o) as shown in Figure A3-1. This intercept point can be expressed in terms of the endpoints of the two chords by the equations

$$y_o = \frac{y_1 - y_2}{x_1 - x_2} (x_o - x_2) + y_2 , \qquad (A3-1)$$

and

$$y_o = \frac{y_3}{x_3} x_o . \qquad (A3-2)$$

Equation (A3-1) relates y_o to x_o in terms of the chord ℓ_1, connecting the point (x_1, y_1) to (x_2, y_2). Equation (A3-2) relates the coordinates in terms of the chord ℓ_2, from $(0, 0)$ to (x_3, y_3). These two equations may be solved for x_o.

$$x_o = \left(\frac{y_3}{x_3} - \frac{y_1 - y_2}{x_1 - x_2} \right)^{-1} \left(y_2 - \frac{y_1 - y_2}{x_1 - x_2} x_2 \right) . \qquad (A3-3)$$

Consider the x-z plane projection of these two chords ℓ_1 and ℓ_2. There will be a point on the z-x projection of the chord ℓ_1 whose x-coordinate values is x_o. Let the z-coordinate value associated with this point be denoted $z_o^{(1)}$. Similarly denote the z-coordinate value on the chord ℓ_2 at this value of the x-coordinate to be $z_o^{(2)}$. Relationships similar to (A3-1) and (A3-2) can now be written for the x-z projection

$$z_o^{(1)} = \frac{z_1 - z_2}{x_1 - x_2} (x_o - x_2) + z_2 , \qquad (A3-4)$$

$$z_o^{(2)} = \frac{z_3}{x_3} x_o . \qquad (A3-5)$$

According to Lemma 2, the regularity condition is guaranteed to hold if $\ell_1 > \ell_2$ in the z-sense for all possible pairs of chords similar to those of Figure 2.6. From the definition of ordering in the z-sense and equations (A3-4) and (A3-5), it can be seen that an identical criterion is

$$z_o^{(1)} - z_o^{(2)} > 0 \qquad \text{(A3-6)}$$

for all pairs of chords similar to those of Figure 2.6 or, alternatively,

$$z_o^{(1)} - z_o^{(2)} < 0 \qquad \text{(A3-7)}$$

for all such pairs of chords. A more compact way of expressing this notion is that the regularity condition is guaranteed to hold if the term

$$z_o^{(1)} - z_o^{(2)} \qquad \text{(A3-8)}$$

is of consistent sign for all possible pairs of chords similar to those of Figure 2.6. This lemma will now be proven by demonstrating that (A3-8) is consistent in sign when (2.14) is consistent in sign.

Substitute (A3-4) and (A3-5) into (A3-8). This substitution yields the term

$$\frac{z_1 - z_2}{x_1 - x_2} (x_o - x_2) + z_2 - \frac{z_3}{x_3} x_o . \qquad \text{(A3-9)}$$

Substituting (A3-3) into (A3-9) yields

$$\frac{z_1 - z_2}{x_1 - x_2} \left[\left(\frac{y_3}{x_3} - \frac{y_1 - y_2}{x_1 - x_2} \right)^{-1} \left(y_2 - \frac{y_1 - y_2}{x_1 - x_2} \right) - x_2 \right] + z_2$$

$$- \frac{z_3}{x_3} \left[\left(\frac{y_3}{x_3} - \frac{y_1 - y_2}{x_1 - x_2} \right)^{-1} \left(y_2 - \frac{y_1 - y_2}{x_1 - x_2} \right) x_2 \right] .$$

$$\text{(A3-10)}$$

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The x-y plane projection of the curve $\mathscr{C}$ is convex by assumption. It is easily seen that one implication of this convexity is that the term

$$\frac{y_3}{x_3} - \frac{y_1 - y_2}{x_1 - x_2} \tag{A3-11}$$

from (A3-10) is consistent in sign. Furthermore, x_2 is consistent in sign because of the convexity and the relabeling of the axes. Therefore, (A3-10) can be multiplied by (A3-11) and divided by x_2 without effecting the sign consistency. Completing these operations, reorganizing and eliminating irrelevent terms yields

$$\left(\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_2}{x_2}\right)\left(\frac{z_1 - z_2}{x_1 - x_2} - \frac{z_3}{x_3}\right) - \left(\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_3}{x_3}\right)\left(\frac{z_1 - z_2}{x_1 - x_2} - \frac{z_2}{x_2}\right).$$

$$\tag{A3-12}$$

Adding and subtracting $\dfrac{z_2}{x_2}$ to the second term of (A3-12) and $\dfrac{y_2}{x_2}$ to the third, expanding and then simplifying the result gives the final form of the term.

$$\left(\frac{y_3}{x_3} - \frac{y_2}{x_2}\right)\left(\frac{z_1}{x_1} - \frac{z_2}{x_2}\right) - \left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)\left(\frac{z_3}{x_3} - \frac{z_2}{x_2}\right) \tag{A3-13}$$

Since there was no operation that would affect the term's consistency in sign, if (A3-13) is consistent in sign, then so is (A3-8). But if (A3-8) is consistent in sign, Lemma 2 can be invoked to conclude that the regularity condition holds. Therefore, if (A3-13) is consistent in sign, and the other conditions of the statement of Lemma 3 hold, the regularity condition is guaranteed.

Q.E.D.

Appendix 4

Proof of Lemma 4:

The proof of this lemma will consist of a demonstration that if the conditions of Lemma 4 are met, then the conditions of Lemma 3 will be met. This being true, Lemma 3 can then be invoked to guarantee that the regularity condition will hold.

From the statement of Lemma 4, the x-y projection of the curve $\mathscr{C}$ is convex. Select a pair of chords, label their end points, and relabel the axes as in Lemma 3. The situation under consideration is now similar to that of Figure A3-1. Referencing this figure will be helpful in understanding the argument that follows.

Consider the term (2.14). For ease of reference (2.14) is reproduced below as (A4-1).

$$\left(\frac{y_3}{x_3} - \frac{y_2}{x_2} \right) \left(\frac{z_1}{x_1} - \frac{z_2}{x_2} \right) - \left(\frac{y_1}{x_1} - \frac{y_2}{x_2} \right) \left(\frac{z_3}{x_3} - \frac{z_2}{x_2} \right) \tag{A4-1}$$

By refering to Figure A3-1, it can be seen that for any particular pair of chords ℓ_1 and ℓ_2

$$x_3 \epsilon \, \mathscr{R}_{x_3}, \text{ where } \mathscr{R}_{x_3} = (x_2, \, x_1) \tag{A4-2}$$

Now if the chord ℓ_1 is assumed to be fixed in position, since

$$y_3 = y(x_3) \tag{A4-3}$$

and

$$z_3 = z(x_3), \tag{A4-4}$$

the term (A4-1) can be thought of as a function of x_3, where x_3 can take values in the range $\mathscr{R}_{x_3}$. According to the statement of Lemma 3, the regularity condition is guaranteed to hold when $y(x)$ is convex and (A4-1) is consistent in sign for all possible chord pairs ℓ_1 and ℓ_2. From the

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present development, it is clear that an equivalent statement is that the regularity condition will be guaranteed when $y(x)$ is convex and (A4-1) is consistent in sign for all x_3 in the range $\mathcal{R}_{x_3}$ for every possible chord ℓ_1.

For a function to be consistent in sign, it must have no zeros or sign changes over its domain. By inspection, (A4-1) will have zeros at $x_3 = x_1$ and at $x_3 = x_2$. If (A4-1) is to be consistent in sign, it can have no zeros between these two points. However, a convex function can have at most two zeros. If (A4-1) is a convex function of x_3 over the region

$$\bar{\mathcal{R}}_{x_3} = [x_2, \ x_1] \tag{A4-5}$$

then the two allowed zeros will be at x_2 and x_1, and the open region $\mathcal{R}_{x_3}$ will contain no zeros. Thus, the term (A4-1) will be consistent in sign if it is convex in x_3 over the region $\bar{\mathcal{R}}_{x_3}$ for every particular chord ℓ_1.

The term (A4-1) will be *strictly convex* over $\bar{\mathcal{R}}_{x_3}$ if the second derivative of (A4-1) with respect to x_3 is consistent in sign over $\bar{\mathcal{R}}_{x_3}$. This second derivative can be written as

$$\left(\frac{y_3}{x_3}\right)'' \left(\frac{z_1}{x_1} - \frac{z_2}{x_2}\right) - \left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right) \left(\frac{z_3}{x_3}\right)''. \tag{A4-6}$$

Rewrite (A4-6) in terms of Taylor series expansions based at x_2.

$$\left[\left(\frac{y_2}{x_2}\right)'' + (x_3 - x_2)\left(\frac{y_2}{x_2}\right)''' + \ldots\right] \left[(x_1 - x_2)\left(\frac{z_2}{x_2}\right)' + \frac{(x_1 - x_2)^2}{2!}\left(\frac{z_2}{x_2}\right)'' + \ldots\right]$$

$$- \left[(x_1 - x_2)\left(\frac{y_2}{x_2}\right)' + \frac{(x_1 - x_2)^2}{2!}\left(\frac{y_2}{x_2}\right)'' + \ldots\right] \left[\left(\frac{z_2}{x_2}\right)''\right.$$

$$\left. + (x_3 - x_2)\left(\frac{z_2}{x_2}\right)''' + \ldots\right] \tag{A4-7}$$

For any particular value of x_3, let $0 < \delta < 1$ such that

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$$\delta(x_1 - x_2) = (x_3 - x_1) \tag{A4-8}$$

The term (A4-7) can be rewritten as

$$(x_1 - x_2) \left\{ \left[\left(\frac{y_2}{x_2}\right)'' + \delta(x_1 - x_2)\left(\frac{y_2}{x_2}\right)''' + \ldots \right] \left[\left(\frac{z_2}{x_2}\right)' + \frac{(x_1 - x_2)}{2!}\left(\frac{z_2}{x_2}\right)'' + \ldots \right] \right.$$
$$\left. - \left[\left(\frac{y_2}{x_2}\right)' + \frac{(x_1 - x_2)}{2!}\left(\frac{y_2}{x_2}\right)'' + \ldots \right] \left[\left(\frac{z_2}{x_2}\right)'' + \delta(x_1 - x_2)\left(\frac{z_2}{x_2}\right)''' + \ldots \right] \right\}. \tag{A4-9}$$

The leading multiplicative term $(x_1 - x_2)$ in (A4-9) is always positive by construction and will have no effect on (A4-9) being consistent in sign. Therefore, it may be discarded.

The curve $\mathscr{C}$, defined parametrically by $y(x)$ and $z(x)$ is smooth by assumption. Therefore, all the derivatives in (A4-9) are well behaved for any choice of the chord ℓ_1. For any particular value of δ in the range $(0, 1)$, let ℓ_1 become "short". That is, let $(x_1 - x_2) \to 0$. Then for any particular "short" chord ℓ_1, (A4-9) becomes (ignoring the leading positive term)

$$\left(\frac{y_2}{x_2}\right)''\left(\frac{z_2}{x_2}\right)' - \left(\frac{y_2}{x_2}\right)'\left(\frac{z_2}{x_2}\right)'' \tag{A4-10}$$

Note that (A4-10) and (2.15) are identical except for the form of their arguments. Thus, a demonstration that consistency in sign for (A4-10) as x_2 varies over all possible values implies that (A4-1) is consistent in sign, will prove the lemma.

Consider the family of all possible chords ℓ_1 of incremental length ("short" chords) that can be defined along the curve $\mathscr{C}$. The final step in this proof is to show that if (A4-10) is consistent in sign as the parameter x_2 varies over all members of this family, then (A4-2) will be consistent in sign for all chord pairs. This can be shown with a "bootstrap" argument.

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Consider the illustration of Figure A4-1.   The illustration shows a convex curve $F(v)$ and three chords joining points on the curve. The nature of the argument will be seen to be uneffected by the sense of the convexity of $F(v)$.

$F(v)$

v_1 $\dfrac{v_1 + v_2}{2}$ v_2 $\dfrac{v_2 + v_3}{2}$ v_3 v
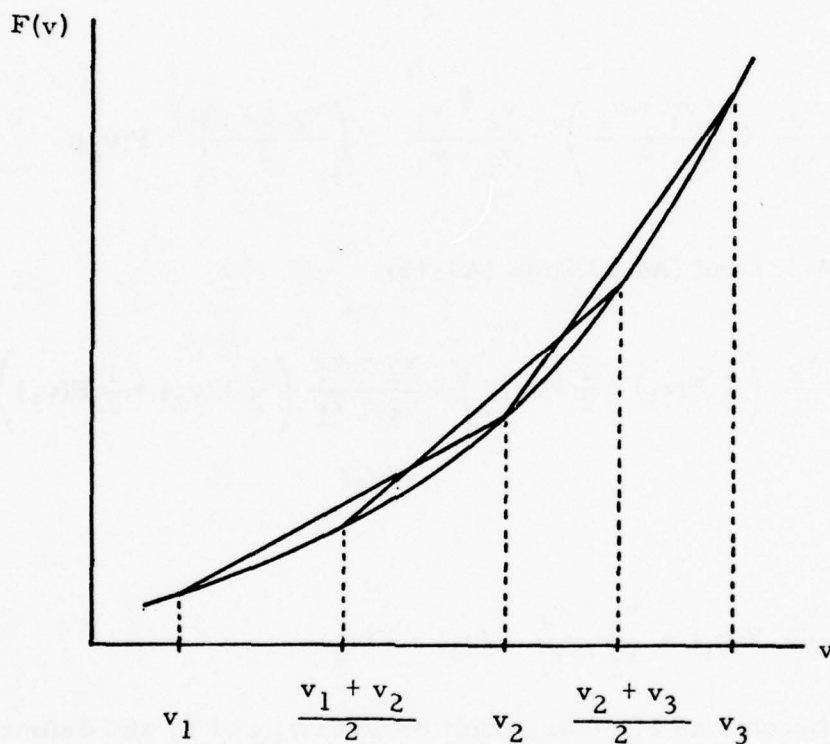
Figure A4-1

By definition, a function $F(\cdot)$ is convex $(\cup)$ in the region $(v_1,\ v_2)$ if for any $\lambda \varepsilon (0,\ 1)$

$$\lambda F(v_1) + (1 - \lambda)\, F(v_2) > F(\lambda v_1 + (1 - \lambda) v_2) \tag{A4-11}$$

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That is, any point on the chord joining two points of the curve is above the corresponding point on the curve. Then if the curve $F(v)$ is known to be convex in each of the regions (v_1, v_2), $\left(\dfrac{v_1 + v_2}{2}, \dfrac{v_2 + v_3}{2}\right)$, and (v_2, v_3), then (A4-11) implies

$$\frac{1}{2} F(v_1) + \frac{1}{2} F(v_2) > F\left(\frac{v_1 + v_2}{2}\right) \qquad (A4-12)$$

$$\frac{1}{2} F(v_2) + \frac{1}{2} F(v_3) > F\left(\frac{v_2 + v_3}{2}\right) \qquad (A4-13)$$

and

$$\frac{v_3 - v_2}{v_3 - v_1} F\left(\frac{v_1 + v_2}{2}\right) + \frac{v_2 - v_1}{v_3 - v_1} F\left(\frac{v_2 + v_3}{2}\right) > F(v_2). \qquad (A4-14)$$

Substituting (A4-12) and (A4-13) into (A4-15)

$$\frac{v_3 - v_2}{v_3 - v_1} \left(\frac{1}{2} F(v_1) + \frac{1}{2} F(v_2)\right) + \frac{v_2 - v_1}{v_3 - v_1} \left(\frac{1}{2} F(v_2) + \frac{1}{2} F(v_3)\right)$$
$$> F(v_2) \qquad (A4-15)$$

or

$$\frac{v_3 - v_2}{v_3 - v_1} F(v_1) + \frac{v_2 - v_1}{v_3 - v_1} F(v_3) > F(v_2) \qquad (A4-16)$$

Allowing v_2 to become an arbitrary point between v_1 and v_3 and defining λ as

$$\lambda = \frac{v_3 - v_2}{v_3 - v_1} = \left(1 - \frac{v_2 - v_1}{v_3 - v_1}\right) \qquad (A4-17)$$

yields the relation

$$\lambda F(v_1) + (1 - \lambda) F(v_3) > F(\lambda v_1 + (1 - \lambda) v_3). \qquad (A4-18)$$

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But then this implies that if the curve is convex in the same sense on three overlapping regions, it is convex on the union of these regions.

Consider the chords shown in Figure A4-1 to be examples of "short" ℓ_1 chords that have been magnified, and the curve of Figure A4-1 to be a magnified section of (A4-1) as a function of x_3 for three sets of (x_1, x_2). If (A4-10) is of the same sign for all three of these values of x_2, then the curve is convex in the same sense under each of these chords separately. But then by the development that leads to (A4-18), the curve is convex under all of these chords combined. But then, by the straight forward extension of these ideas, if (A4-10) is consistent in sign for all incremental chords along the curve $\mathscr{C}$, then (A4-6) will be consistent in sign for any particular chord ℓ_1. But this guarantees that (A4-1) will be consistent in sign for all pairs of chords ℓ_1 and ℓ_2. Thus, if (A4-10) is consistent in sign for all points on the curve $\mathscr{C}$, and $y(x)$ is convex, then the criteria of Lemma 3 are satisfied, and the regularity condition is guaranteed to hold.   Q. E. D.

Proof of Corollary 4.1:

The corollary is an immediate consequence of the lemma for the special case that the function $y(x)$ is quadratic. Plugging the quadratic

$$y(x) = x^2 + ax \tag{A4-19}$$

into (2.15) yields

$$- \left(\frac{z(x)}{x} \right)'' . \tag{A4-20}$$

But the property of consistency in sign is independent of what the sign actually is. Therefore, the minus sign may be dropped. Q.E.D.

Proof of Corollary 4.2:

Corollary 4.1 stated that for the special case where the function $y(x)$ is quadratic, that if $\left(\frac{z(x)}{x} \right)''$ is consistent in sign, the regularity conidition will hold. But from calculus

$$\left(\frac{z(x)}{x} \right)'' = \frac{x^2 z''(x) - 2(xz'(x) - z(x))}{x^3} . \tag{A4-21}$$

As before, it can be assumed that the origin has been positioned such that $x \geq 0$. Therefore, the denominator of (A4-21) plays no part in telling whether or not the term is consistent in sign.

Consider the term

$$\int x^2 z'''(x)dx . \tag{A4-22}$$

Clearly if $z'''(x)$ is consistent in sign, (A4-22) will be consistent in sign. Integrating by parts yields

$$\int x^2 z'''(x)dx = x^2 z''(x) - 2\int x z''(x)dx =$$
$$= x^2 z''(x) - 2(xz'(x) - z(x)). \tag{A4-23}$$

But the right hand side of (A4-23) is identical to the numerator of (A4-21). Therefore, if $z'''(x)$ is consistent in sign, $\left(\dfrac{z(x)}{x}\right)''$ will be consistent in sign. Thus Corollary 4.1 may be invoked, and the regularity condition holds.

$$\text{Q.E.D.}$$

Appendix 5

Proof of Lemma 5:

Consider a chord that lies on the surface of the convex hull $\mathscr{H}$ generated by a smooth twisted curve $\mathscr{C}$. Denote the end points of the chord as p_1 and p_2. The points p_1 and p_2 are clearly points of the curve $\mathscr{C}$. Consider a second chord connecting point p_1 to point p_3, a point on the curve $\mathscr{C}$ that is in the vicinity of p_2. These two chords are segments of intersecting lines and, therefore, define a plane. Consider the sequence of planes formed by allowing p_3 to approach p_2 as a limit along $\mathscr{C}$. Clearly the limiting plane is the tangent plane to $\mathscr{H}$ whose characteristic is the chord (p_1, p_2). But by definition [24] the tangent line to $\mathscr{C}$ at p_2 will be the limiting position of the chord (p_2, p_3), which is clearly in the plane. Thus, the tangent line and tangent plane approach their limiting positions together, and the line is in the plane. But this limiting argument can be applied at either end of the original chord (P_1, P_2). Thus, the tangent lines at both ends of the chord must lie in the tangent plane. Q.E.D.

Appendix 6

Proof of Lemma 6:

Given two points, $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ on the curve $\mathscr{C}$, consider a point p_3 on the tangent line to $\mathscr{C}$ at p_1. The tangent line at p_1 is given by equation (3.1) to be

$$X - x_1 = \frac{Y - y_1}{y_1'} = \frac{Z - z_1}{z_1'} \qquad (A6.1)$$

For simplicity, let

$$x_3 = x_1 + 1. \qquad (A6.2)$$

Then from (A6.1)

$$y_3 = y_1 + y_1' \qquad (A6.3)$$

$$z_3 = z_1 + z_1'. \qquad (A6.4)$$

Similarly there will be a point p_4 on the tangent line to $\mathscr{C}$ at p_2 such that

$$x_4 = x_2 + 1 \qquad (A6.5)$$

$$y_4 = y_2 + y_2' \qquad (A6.6)$$

$$z_4 = z_2 + z_2'. \qquad (A6.7)$$

The statement of Lemma 5 says that the tangent lines to the curve $\mathscr{C}$ at each end point of a chord that is on the surface of the convex hull $\mathscr{H}$ generated by $\mathscr{C}$, must both be in the tangent plane to $\mathscr{H}$ that has the chord as its characteristic. In the present context, another way to state this is to say that the plane defined by the points p_1, p_2 and p_3 must be identical to the plane defined by the points p_1, p_2, and p_4. If the chord (p_1, p_2) is a surface chord, this plane must also be a tangent plane to $\mathscr{H}$.

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A plane in E^3 can be described by the equation [24]

$$aX + bY + cZ = 1 \qquad (A6.8)$$

where X, Y, and Z are the coordinates of a point of the plane, and a, b, and c are the specifying coefficients of the plane. The coefficient values for the plane defined by the points p_1, p_2, and p_3 may be determined by solving the matrix equation

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (A6.9)$$

or

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (A6.10)$$

However, the necessary condition states that the plane defined by (p_1, p_2, p_3) must be identical to the plane defined by (p_1, p_2, p_4). The plane defined by (p_1, p_2, p_4) will have the coefficients

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad (A6.11)$$

If these planes are identical,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \qquad (A6.12)$$

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where the equality is on an element by element basis. Equations (A6.10) -
(A6.12) can be combined to yield

$$
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (A6.13)
$$

or

$$
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} . \quad (A6.14)
$$

The inverse matrix in (A6.14) can be computed by the method of cofactors
[25] to be

$$
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} = \frac{1}{D} \begin{bmatrix} (y_2 z_3 - y_3 z_2) & -(y_1 z_3 - y_3 z_1) & (y_1 z_2 - y_2 z_1) \\ -(x_2 z_3 - x_3 z_2) & (x_1 z_3 - x_3 z_1) & -(x_1 z_2 - x_2 z_1) \\ (x_2 y_3 - x_3 y_2) & -(x_1 y_3 - x_3 y_1) & (x_1 y_2 - x_2 y_1) \end{bmatrix} \quad (A6.15)
$$

where

$$
D = x_1 y_2 z_3 + x_2 y_3 z_1 + x_3 y_1 z_2 - x_1 y_3 z_2 - x_2 y_1 z_3 - x_3 y_2 z_1 \quad (A6.16)
$$

Equation (A6.15) can be combined with equation (A6.14), and the matrix
multiplications carried out. This will yield three equations in the coordinates
of points p_1, p_2, p_3, and p_4. Upon collection of terms, two of the three
equations will be seen to be identities. The remaining equation is

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$$x_4[(y_2 z_3 - y_3 z_2) - (y_1 z_3 - y_3 z_1) + (y_1 z_2 - y_2 z_1)]$$

$$+ y_4 \ [-(x_2 z_3 - x_3 z_2) + (x_1 z_3 - x_3 z_1) - (x_1 z_2 - x_2 z_1)]$$

$$+ z_4 \ [(x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)] = D$$

$$(A6.17)$$

Combining equations (A6.2) - (A6.7) with (A6.17) yields

$$[y_1'(z_1 - z_2) - z_1'(y_1 - y_2)]$$

$$+ y_2'[-(z_1 - z_2) + z_1'(x_1 - x_2)]$$

$$+ z_2'[(y_1 - y_2) - y_1'(x_1 - x_2)] = 0 \qquad (A6.18)$$

Rearranging the variables gives

$$\frac{y_1 - y_2}{x_1 - x_2}(z_1' - z_2') - \frac{z_1 - z_2}{x_1 - x_2}(y_1' - y_2') + (y_1' z_2' - y_2' z_1') = 0 \qquad (A6.19)$$

which is the form required for Lemma 6. Thus, requiring that both tangent lines at the endpoints of a chord be in the tangent plane to $\mathcal{H}$ whose surface characteristic is the chord (statement of Lemma 5) is equivalent to equation (A6.19) being true. Therefore, equation (A6.19) is a necessary condition for the chord joining points p_1 and p_2 to be a surface chord of $\mathcal{H}$.

Q.E.D.

Proof of Corollary 6.1:

Substituting the relations

$$y_i = x_i^2 + ax_i \qquad i = 1, 2 \qquad (A6.20)$$

$$y_i' = 2x_i + a \qquad i = 1, 2 \qquad (A6.21)$$

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into equation (A6.19) and simplifying the result yields

$$\frac{z_1' + z_2'}{2} - \frac{z_1 - z_2}{x_1 - x_2} = 0 \qquad\qquad \text{(A6.22)}$$

which is the required relationship. Q.E.D.

Proof of Corollary 6.2:

For any pair of points as in Corollary 6.1 where $x_1 > x_2$, shift the axis such that $p_2 = (x_2, y_2, z_2) = (0, 0, 0)$. Then equation (A6.22) can be written as

$$\frac{z_1' + z_2'}{2} - \frac{z_1}{x_1} = 0 \qquad\qquad \text{(A6.23)}$$

Consider the L.H.S. of equation (A6.23) as a function in x_1,

$$\mathscr{F}(x_1) = \frac{z'(x_1) + z_2'}{2} - \frac{z(x_1)}{x_1} \qquad\qquad \text{(A6.24)}$$

for the appropriate constant z_2'. Clearly the function $\mathscr{F}(x_1)$ will approach zero as $x_1 \to x_2 = 0$. A sufficient condition insuring that $\mathscr{F}(x_1)$ will have no roots for $x_1 > 0$ is that the first derivative of $\mathscr{F}(x_1)$ with respect to x_1 be consistent in sign. Thus, the condition is that

$$\mathscr{F}'(x_1) = \frac{z_1''}{2} - \frac{x_1 z_1' - z_1}{x_1^2} \qquad\qquad \text{(A6.25)}$$

be consistent in sign. But rearranging (A6.25)

$$\mathscr{F}'(x_1) = \frac{x_1^2 z_1'' - 2(x_1 z_1' - z_1)}{2x_1^2}$$

$$= \frac{x_1}{2} \left(\frac{z(x_1)}{x_1} \right)'' . \qquad\qquad \text{(A6.26)}$$

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But $x_1 > 0$ by construction, and so will not effect the consistency of the sign. Thus, an equivalent condition is that $\left(\dfrac{z(x)}{x}\right)''$ be of consistent sign. Q.E.D.

Proof of Corollary 6.3:

The necessary condition for a chord joining points p_1 and p_2 on the twisted curve of Corollary 6.1 to be a surface chord of convex hull is given as equation (A6.22). Consider the Taylor series expansions about p_1. Then

$$z_2 = z_1 + (x_2 - x_1)z_1' + \frac{(x_2 - x_1)^2}{2!} z_1'' + \frac{(x_2 - x_1)^3}{3!} z_1''' + \cdots \qquad (A6.27)$$

$$z_2' = z_1' + (x_2 - x_1)z_1'' + \frac{(x_2 - x_1)^2}{2!} z_1''' + \frac{(x_2 - x_1)^3}{3!} z_1^{(iv)} + \cdots \qquad (A6.28).$$

Combining (A6.27) and (A6.28) with (A6.22) and simplifying yields

$$\frac{z_1' + z_2'}{2} - \frac{z_1 - z_2}{x_1 - x_2} = \left(\frac{1}{2(2!)} - \frac{1}{3!}\right)(x_2 - x_1)^2 z_1''' + \left(\frac{1}{2(3!)} - \frac{1}{4!}\right)(x_2 - x_1)^3 z^{(iv)}$$
$$+ \cdots \qquad (A6.29)$$

For short chords (i.e., $|x_2 - x_1|$ is small), the only term of significant size will be the term involving z_1'''. Thus, for the right hand side of (A6.29) to equal zero for short chords, it is necessary that

$$z''' = 0 \qquad (A6.30)$$

The solutions to (A6.30) will then be the possible central points. Q.E.D.

Appendix 7

Proof of Lemma 7:

Lemma 5 establishes that a necessary condition for a chord to be a surface chord of the convex hull generated by twisted curve $\mathscr{C}$, is that the chord and the tangent lines to $\mathscr{C}$ at both end points of the chord must all be in the same plane. Lemma 6 states this necessary condition in terms of an equation in the coordinates of the endpoints of the chord. The ideas of Lemma 5 and the methods and equations of Lemma 6 can be used to generate a set of necessary conditions for when a planar section can be a feature of the surface of the convex hull generated by $\mathscr{C}$.

A planar section is defined by three points of the curve $\mathscr{C}$, to be denoted $p_1 = (x_1, y_1, z_1)$, $p_2 = (x_2, y_2, z_2)$, and $p_3 = (x_3, y_3, z_3)$. Equivalently, the section could be defined by the three chords joining these points and forming a triangle. These chords are clearly in the plane, and will be surface chords if the planar section is on the surface. From Lemma 6, a necessary condition for the chord (p_1, p_2) to be on the surface of the convex hull is that the equation

$$\frac{y_1 - y_2}{x_1 - x_2} (z_1' - z_2') - \frac{z_1 - z_2}{x_1 - x_2} (y_1' - y_2') + (y_1' z_2' - y_2' z_1') = 0 \qquad \text{(A7.1)}$$

be satisfied. It can be seen that equation (A7.1) is equivalent to equation (3.7) in the statement of Lemma 7. Similarly, equations (3.8) and (3.9) can be seen to correspond to the necessary conditions for the chords (p_1, p_3) and (p_2, p_3) to be surface chords respectively. Thus, equations (3.7) - (3.9) are the necessary conditions for the three chords (p_1, p_2), (p_1, p_3), and (p_2, p_3) to be surface chords. It is clearly necessary for these chords to be surface chords for the planar section (p_1, p_2, p_3) to lie on the surface of the convex hull.

A second set of necessary conditions can be imposed on the planar section (p_1, p_2, p_3). For a smooth twisted curve $\mathscr{C}$, there is a unique relationship between surface characteristics of a convex hull, and the tangent planes to the convex hull. If this were not true, the derivative of the curve $\mathscr{C}$ would not be uniquely defined, and the curve would not be smooth. This unique relationship means that all three chords, and the three tangent lines to the curve $\mathscr{C}$ at the points p_1, p_2, and p_3, must all lie in the same plane. Thus, the tangent line to the curve $\mathscr{C}$ at the point p_2 must be in the plane defined by the chords (p_1, p_2) and (p_2, p_3). Similarly, the tangent line at p_1 must be in the plane defined by the chords (p_1, p_2) and (p_1, p_3), and the tangent line at p_3 in the plane defined by the chords (p_1, p_3) and (p_2, p_3).

A plane in E^3 is defined by the equation

$$aX + bY + cZ = 1 \tag{A7.2}$$

where (X, Y, Z) is any point in the plane, and a, b, and c are the defining coefficients of the plane. For the plane defined by the points (p_1, p_2, p_3), these coefficients are the solution to the matrix equation,

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{A7.3}$$

or

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_3 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{A7.4}$$

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Following the development presented in Appendix 6, select points p_4, p_5, and p_6 on the tangent lines at points p_1, p_2, and p_3 respectively. From the definition of the tangent line, select these points such that

$$x_{i+3} = x_i + 1 \qquad (A7.5)$$

$$y_{i+3} = y_i + y_i' \qquad (A7.6)$$

$$z_{i+3} = z_i + z_i' \qquad (A7.7)$$

where $i \in (1, 2, 3)$, and the primes denote differentiation by x_i.

Combining equations (A7.3) - (A7.7) with the statement of the set of necessary conditions yields the three matrix equations

$$
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_1+1 & y_1+y_1' & z_1+z_1' \end{bmatrix}
\cdot
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1}
\cdot
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\qquad (A7.8)
$$

$$
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_2+1 & y_2+y_2' & z_2+z_2' \end{bmatrix}
\cdot
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1}
\cdot
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\qquad (A7.9)
$$

$$
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_3+1 & y_3+y_3' & z_3+z_3' \end{bmatrix}
\cdot
\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}^{-1}
\cdot
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\qquad (A7.10)
$$

As was the case in the development in Appendix 6, the first two of the three equations in each of the matrix equations (A7.8) - (A7.10) will be simple identities. The third equation in each of (A7.8) - (A7.10) can be simplified and rearranged and rewritten as the matrix expression

$$
\begin{bmatrix}
1 & y_1' & z_1' \\
1 & y_2' & z_2' \\
1 & y_3' & z_3'
\end{bmatrix}
\cdot
\begin{bmatrix}
y_3(z_1 - z_2) - z_3(y_1 - y_2) + (y_1 z_2 - y_2 z_1) \\
-x_3(z_1 - z_2) + z_3(x_1 - x_2) - (x_1 z_2 - x_2 z_1) \\
x_3(y_1 - y_2) - y_3(x_1 - x_2) + (x_1 y_2 - x_2 y_1)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\qquad (A7.11)
$$

Equation (A7.11) is the exact form of equations (3.4) - (3.6). Thus it has been demonstrated that it is necessary for three points (p_1, p_2, p_3) of the curve $\mathscr{C}$ to satisfy both equation (A7.11) and the three equations of the form (A7.1) in order to be a surface feature of the convex hull generated by $\mathscr{C}$.

Q.E.D.

Proof of Corollary 7.1

For the twisted curve $\mathscr{C}$ of Lemma 7, let $y(x) = x^2 + ax$ for any real number a. Equations (3.7) - (3.9) will simplify to equations (3.13) - (3.15). This is exactly the same simplification that takes place in demonstrating Corollary 6.1 from Lemma 6.

Combining the relation $y(x) = x^2 + ax$ with equation (A7.11) yields

$$
\begin{bmatrix}
1 & 2x_1 + a & z_1' \\
1 & 2x_2 + a & z_2' \\
1 & 2x_3 + a & z_3'
\end{bmatrix}
\cdot
\begin{bmatrix}
(x_3^2 + ax_3)(z_1 - z_2) - z_3(x_1^2 + ax_1 - x_2^2 - ax_2) \\
\qquad\qquad + ((x_1^2 + ax_1)z_2 - (x_2^2 + ax_2)z_1) \\
-x_3(z_1 - z_2) - z_3(x_1 - x_2) - (x_1 z_2 - x_2 z_1) \\
x_3(x_1^2 - x_2^2) - x_3^2(x_1 - x_2) + (x_1 x_2^2 - x_2 x_1^2)
\end{bmatrix}
$$

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$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(A7.12)

Multiplying equation (A7.12) out, simplifying, and rearranging yields the three equations

$$(x_1 - x_3) \frac{z_1 - z_2}{x_1 - x_2} - (x_1 - x_2) \frac{z_1 - z_3}{x_1 - x_3} - z_1' (x_2 - x_3) = 0 \qquad (A7.13)$$

$$(x_2 - x_3) \frac{z_1 - z_2}{x_1 - x_2} + (x_1 - x_2) \frac{z_2 - z_3}{x_2 - x_3} - z_2' (x_1 - x_3) = 0 \qquad (A7.14)$$

$$(x_1 - x_3) \frac{z_2 - z_3}{x_2 - x_3} - (x_2 - x_3) \frac{z_1 - z_3}{x_1 - x_3} - z_3' (x_1 - x_2) = 0 \qquad (A7.15)$$

The three equations (A7.13) - (A7.15) match the equations (3.10) - (3.12) of Corollary 7.1. Thus the six equations in Corollary 7.1 are necessary conditions by Lemma 7 under the restriction that $y(x) = x^2 + ax$ for any real number a.

Q.E.D.

Appendix 8

Proof of Lemma 8

According to Corollary 7.1, in order for three points on a smooth twisted curve $\mathscr{C}$ to define a planar section on the surface of the convex hull generated by $\mathscr{C}$, when $\mathscr{C}$ is defined parametrically in x by $y(x) = x^2 + ax$, $z(x)$, for some real number a, it is necessary for the coordinate values of the three points to satisfy the six equations

$$(x_1 - x_3) \frac{z_1 - z_2}{x_1 - x_2} - (x_1 - x_2) \frac{z_1 - z_3}{x_1 - x_3} - z_1' (x_2 - x_3) = 0 \qquad \text{(A8.1)}$$

$$(x_2 - x_3) \frac{z_1 - z_2}{x_1 - x_2} - (x_1 - x_2) \frac{z_2 - z_3}{x_2 - x_3} - z_2' (x_1 - x_3) = 0 \qquad \text{(A8.2)}$$

$$(x_1 - x_3) \frac{z_2 - z_3}{x_2 - x_3} - (x_2 - x_3) \frac{z_1 - z_3}{x_1 - x_3} - z_3' (x_1 - x_2) = 0 \qquad \text{(A8.3)}$$

$$\frac{z_1' + z_2'}{2} - \frac{z_1 - z_2}{x_1 - x_2} = 0 \qquad \text{(A8.4)}$$

$$\frac{z_1' + z_3'}{2} - \frac{z_1 - z_3}{x_1 - x_3} = 0 \qquad \text{(A8.5)}$$

$$\frac{z_2' - z_3'}{2} - \frac{z_2 - z_3}{x_2 - x_3} = 0 \qquad \text{(A8.6)}$$

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Equations (A8.1) - (A8.6) can be treated as six linear equations

in the six unknowns z_1', z_2', z_3', $\frac{z_1 - z_2}{x_1 - x_2}$, $\frac{z_1 - z_3}{x_1 - x_3}$, and $\frac{z_2 - z_3}{x_2 - x_3}$. The

equation set can then be solved in terms of the differences $(x_1 - x_2)$, $(x_1 - x_3)$,
and $(x_2 - x_3)$. Combining equations (A8.1) - (A8.3) with (A8.4) - (A8.6) and
eliminating the unknown ratios, the results can be written as the matrix
equation

$$
\begin{bmatrix} -3 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} (x_2 - x_3)z_1' \\ (x_1 - x_3)z_2' \\ (x_1 - x_2)z_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{A8.7}
$$

But the 3 x 3 coefficient matrix on the left hand side of (A8.7) is not
singular. Therefore, it is invertible. Thus, the solution to (A8.7) is

$$
\begin{bmatrix} (x_2 - x_3)z_1' \\ (x_1 - x_3)z_2' \\ (x_1 - x_2)z_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{A8.8}
$$

But by assumption (Section 2.1), the curve $\mathscr{C}$ is generated by functions
that are smooth and single valued on I_x. Thus, $x_2 \neq x_3$, $x_1 \neq x_3$, and
$x_1 \neq x_2$. Therefore, in order for (A8.8) to be satisfied, it must be
true that

$$
z_1' = z_2' = z_3' = 0. \tag{A8.9}
$$

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However, combining (A8.9) with (A8.4) - (A8.6) yields

$$\frac{z_1 - z_2}{x_1 - x_2} = \frac{z_1 - z_3}{x_1 - x_3} = \frac{z_2 - z_3}{x_2 - x_3} = 0. \tag{A8.10}$$

But (A8.10) implies that

$$z_1 = z_2 = z_3, \tag{A8.11}$$

which clearly violates the assumed single valuedness of the function $z(x)$. Therefore, (A8.11) cannot hold, and no three interior points of $\mathcal{C}$ can define a planar section on the surface of the convex hull generated by $\mathcal{H}$.

Q.E.D.

It is noted that in the somewhat more general case where $z(x)$ is not required to be single valued on I_x, it can occur that equations (A8.9) - (A8.11) are satisfied for three points on the curve $\mathcal{C}$. When this occurs it is easy to determine if the plane is on the surface of the convex hull. From (A8.11) it is clear that the plane in question is parallel to the x-y plane. Thus, in order for the plane to be on the surface, z_1 must be either the largest or the smallest value of $z(x)$ on I_x. If this is not true, there must be points of the curve $\mathcal{C}$ both above and below the plane in the z-sense, and the plane could not be on the surface.

Appendix 9

Proof of Lemma 9:

Consider a smooth finite length twisted curve $\mathscr{C}$, given parametrically in x by $y(x) = x^2 + ax$ and $z(x)$, where a is any real number. Denote the end points of $\mathscr{C}$ by $p_1 = (x_1, y_1, z_1)$ and $p_3 = (x_3, y_3, z_3)$, and let $p_2 = (x_2, y_2, z_2)$ be an interior point of $\mathscr{C}$. Consider the plane defined by the points (p_1, p_2, p_3). From Corollary 7.1 it is known that a necessary condition that must be satisifed if the plane (p_1, p_2, p_3) is a surface plane of the convex hull $\mathscr{H}$, generated by $\mathscr{C}$ is that

$$(x_2 - x_3) \cdot \frac{z_1 - z_2}{x_1 - x_2} + (x_1 - x_2) \frac{z_2 - z_3}{x_2 - x_3} - z_2'(x_1 - x_3) = 0. \tag{A9.1}$$

Relabel the x and z axis so that $x_1 = z_1 = 0$. Then (A9.1) becomes

$$(x_2 - x_3) \frac{z_2}{x_2} - x_2 \frac{z_2 - z_3}{x_2 - x_3} + z_2' x_3 = 0, \tag{A9.2}$$

or

$$-x_2 \left(\frac{z_2 - z_3}{x_2 - x_3} - \frac{z_2}{x_2} \right) + x_3 \left(z_2' - \frac{z_2}{x_2} \right) = 0, \tag{A9.3}$$

or

$$\frac{-x_2 x_3}{x_2 - x_3} \left(\frac{z_2}{x_2} - \frac{z_3}{x_3} \right) + x_2 x_3 \left(\frac{x_2 z_2' - z_2}{x_2^2} \right) = 0. \tag{A9.4}$$

But

$$\left(\frac{z(x)}{x} \right)' = \frac{xz' - z}{x^2} . \tag{A9.5}$$

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Therefore,

$$\left(\frac{z_2}{x_2}\right)' = \frac{\left(z_2/x_2 - z_3/x_3\right)}{x_2 - x_3} \; . \tag{A9.6}$$

Thus, (A9.6) is equivalent to (A9.1) for the situation where p_1 has been moved to the origin. It is noted that the other necessary conditions from Corollary 7.1 are not important because the derivatives at the end points of $\mathscr{C}$ (points p_1 and p_3) will not be uniquely defined.   Q.E.D.

Appendix 10

Proof of Lemma 10:

Following the development presented in Appendix 7, it is seen that a planar section defined by the points p_1, p_2, and p_3, where p_1 is taken to be an endpoint of the smooth twisted curve $\mathscr{C}$, must satisfy the following necessary conditions if it is to be on the surface of the convex hull generated by $\mathscr{C}$.

$$(x_2 - x_3) \frac{z_1 - z_2}{x_1 - x_2} + (x_1 - x_2) \frac{z_2 - z_3}{x_2 - x_3} - z_2' (x_1 - x_3) = 0 \qquad (A10.1)$$

$$(x_1 - x_3) \frac{z_2 - z_3}{x_2 - x_3} - (x_2 - x_3) \frac{z_1 - z_3}{x_1 - x_3} - z_3'(x_1 - x_2) = 0 \qquad (A10.2)$$

$$\frac{z_2' + z_3'}{2} - \frac{z_2 - z_3}{x_2 - x_3} = 0 \qquad (A10.3)$$

The other equations in Corollary 7.1 are not required because z_1', the derivative at the end point of the curve $\mathscr{C}$, is not uniquely defined.

Relabel the axis such that p_1 is shifted to the origin. Equations (A10.1) and (A10.2) may now be rewritten as:

$$(x_2 - x_3) \frac{z_2}{x_2} - x_2 \frac{z_2 - z_3}{x_2 - x_3} + z_2' x_3 = 0, \qquad (A10.4)$$

and

$$-x_3 \frac{z_2 - z_3}{x_2 - x_3} - (x_2 - x_3) \frac{z_3}{x_3} + z_3' x_2 = 0. \qquad (A10.5)$$

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In Appendix 9 it was shown that equation (A10.4) is equivalent to

$$\left(\frac{z_2}{x_2}\right)' = \frac{\left(z_2/x_2 - z_3/x_3\right)}{x_2 - x_3} \; . \tag{A10.6}$$

However, it can be seen by inspection that equation (A10.5) is identical in form to equation (A10.4). The only difference is that the indexes 2 and 3 are reversed in location. Thus, equation (A10.5) must be equivalent to

$$\left(\frac{z_3}{x_3}\right)' = \frac{\left(z_3/x_3 - z_2/x_2\right)}{x_3 - x_2} \; . \tag{A10.7}$$

But the right hand side of (A10.6) and (A10.7) are numerically the same. Thus, the necessary condition for a class one planar section becomes

$$\left(\frac{z_2}{x_2}\right)' = \left(\frac{z_3}{x_3}\right)' = \frac{\left(z_3/x_3 - z_2/x_2\right)}{x_3 - x_2} \; . \tag{A10.8}$$

Q. E. D.

REFERENCES

[1]   Lucky, R. W., J. Salz, and E. J. Weldon, Jr., Principles
        of Data Communication, McGraw-Hill, New York, 1968.

[2]   Ho, E. Y., and Y. S. Yeh, "A New Approach for Evaluating
        the Error Probability in the Presence of Intersymbol Interference
        and Additive Gaussian Noise," Bell Syst. Tech. J., Vol. 49,
        pp. 2249-2266, Nov. 1970.

[3]   Shimbo, O. and M. I. Celebiler, "The Probability of Error
        Due to Intersymbol Interference and Gaussian Noise in Digital
        Communication Systems," IEEE Trans. Comm. Tech.,
        Vol. COM-19, pp. 113-119, April 1971.

[4]   Prabhu, V. K., "Some Considerations of Error Bounds in Digital
        Systems," Bell Syst. Tech. J., Vol. 50, pp. 3127-3151,
        Dec. 1971.

[5]   Benedetto, S., G. DeVincentiis, and A. Luvison, "Error
        Probability in the Presence of Intersymbol Interference and
        Additive Noise for Multilevel Digital Systems," IEEE Trans.
        Comm. Tech., Vol. COM-21, pp. 181-188, March 1973.

[6]   Saltzberg, B. R., "Intersymbol Interference Error Bounds
        with Application to Ideal Bandlimited Signaling," IEEE Trans.
        Inform. Theory, Vol. IT-14, pp. 563-568, July 1968.

[7]   Lugannani, R., "Intersymbol Interference and Probability of
        Error in Digital Systems," IEEE Trans. Inform. Theory,
        Vol. IT-15, pp. 682-688, Nov. 1969.

[8]   Glave, F. E., "An Upper Bound on the Probability of Error
        Due to Intersymbol Interference for Correlated Digital Signals,"
        IEEE Trans. Inform. Theory, Vol. IT-18, pp. 356-363, May 1972.

[9]   Matthews, J. W., "Sharp Error Bounds for Intersymbol Interference,"
        IEEE Trans. Inform. Theory, Vol. IT-19, pp. 440-448, July 1973.

[10]   McLane, P. J., "Lower Bounds for Finite Intersymbol Interference
        Error Rates," IEEE Trans. Comm. Tech., Vol. COM-22,
        pp. 853-857, June 1974.

[11]   Yao, K. and R. M. Tobin, "Moment Space Upper and Lower Error
        Bounds for Digital Systems with Intersymbol Interference,"
        IEEE Trans. Inform. Theory, Vol. IT-22, pp. 65-74, Jan. 1976.

[12]   Yan, T. Y., Moment Space Error Bounds for Digital Communication
        Systems with Intersymbol Interference Based on N^{th} Moment,
        M. S. in Engineering, Systems Science Department, University
        of California, Los Angeles, Jan. 1976.

[13]   Yao, K., "Quadratic-Exponential Moment Error Bounds for
        Digital Communication Systems," Conf. Rec. of the Tenth
        Annual Asilomar Conf. on Circuits, Systems, and Computers,
        Pacific Groves, Calif., Nov. 1976, pp. 99-103.

[14]   Yao, K. and R. M. Tobin, "Multi-Dimensional Moment Space
        Error Bounds for Digital Systems with Multiple Interferences,"
        Abstract of Papers of the 1976 Int. Symposium on Information
        Theory, Ronneby, Sweden, June, 1976, p. 69.

[15]   Yao, K. and E. Biglieri, "Moment Inequalities for Error
         Probabilities in Digital Communication Systems," Proc. of
         the Nat. Telecommunication Conf., Los Angeles, Calif.,
         Dec. 1977.

[16]   Tobin, R. M., and K. Yao, "Upper and Lower Error Bounds
         for Coherent Phase-Shift-Keyed (CPSK) Systems with Cochannel
         Interference," IEEE Trans. Comm. Tech., Vol. COM-25,
         pp. 281-287, Feb. 1977.

[17]   Yao, K., "Error Probability of Asynchronous Spread Spectrum
         Multiple Access Communication Systems," IEEE Trans. Comm.
         Tech., Vol. COM-25, pp. 803-809, Aug. 1977.

[18]   Yao, K., "Error Proability of Spread Spectrum Multiple Access
         Communication Systems," Proc. of the 1976 Conf. on Information
         Sciences and Systems, John Hopkins Univ., Baltimore, Maryland,
         March 1976, pp. 67-72.

[19]   Yao, K., Performance Bounds on Spread Spectrum Multiple
         Access Communication Systems," Proc. of the International
         Telemetry Conf., Los Angeles, Calif., Sept. 1976, pp. 317-325.

[20]   Dresher, M., S. Karlin, L. S. Shapley, "Polynomial Games,"
         Contributions to the Theory of Games, Annals of Mathematics
         Studies, No. 24, 1950, pp. 161-180.

[21]   Dresher, M., "Moment Spaces and Inequalities," Duke Math.
         J., Vol. 20, pp. 261-271, June 1953.

[22]   Rockafellar, R. T., Convex Analysis, Princeton University
         Press, Princeton, New Jersey, 1970.

[23]   Balakrishnan, A. V., Applied Functional Analysis, Springer-
         Verlag, New York, 1976.

[24]   Eisenhart, L. P., A Treatise on the Differential Geometry
         of Curves and Surfaces, Dover, New York, 1960.

[25]   Wylie, C. R. Jr., Advanced Engineering Mathematics, 2nd
         Edition, McGraw-Hill, New York, 1960.

[26]   Appel, A., and P. M. Will, "Determining the Three-Dimensional
         Convex Hull of a Polyhedron," IBM J. of Research and Development,
         pp. 590-601, Nov. 1976.

[27]   Preparata, F. P., and S. J. Hong, "Convex Hulls of Finite
         Planar and Spatial Sets of Points," Coordinated Science
         Laboratory, University of Illinois at Urbana-Champaign,
         Urbana, Illinois, Report R-682 (UILU-ENG 75-2217), April, 1975.

[28]   Wilhelmsen, D. R., "A Nearest Point Algorithm for Convex
         Polyhedral Cones and Applications to Positive Linear Approximation,"
         Mathematics of Computation, Vol. 30, No. 133, pp. 48-57,
         January 1976.

[29]   Balinski, M. L., "An Algorithm for Finding all Verticies of
         Convex Polyhedral Sets," J. Soc. Indust. Appl. Math., Vol. 9,
         No. 1, pp. 72-88, March 1961.

[30]   Kendall, M. G., A Course in the Geometry of n Dimensions,
         Charles Griffin and Co. Ltd., London, 1961.

[31]   Grünbaum, B., Convex Polytopes, Interscience Publishers,
         New York, 1967.

[32]   Busemann, H., <u>Convex Surfaces</u>, Interscience Publishers, New York, 1967.

[33]   Lyusternik, L. A., <u>Convex Figures and Polyhedra</u>, D. C. Heath and Co., Boston, 1966.